Physics Letters A 385 (2021) 126979

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

A class of solutions of the two-dimensional Toda lattice equation

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ARTICLE INFO

Article history:
Received 11 September 2020
Received in revised form 15 October 2020
Accepted 23 October 2020
Available online 26 October 2020
Communicated by F. Porcelli

Keywords:
Toda lattice
Solitons
Dispersive nonlinear partial differential equations

ABSTRACT

A method is proposed to systematically generate solutions of the two-dimensional Toda lattice equation in terms of previously known solutions $\phi(x, y)$ of the two-dimensional Laplace's equation. The solitonic solution of Nakamura's [J. Phys. Soc. Jpn. 52 (1983) 380] is shown to correspond to one particular choice of $\phi(x, y)$.

The numerical discovery of solitons in collisionless plasmas [1], accompanied by an explanation for the recurrence of states in nonlinear prototype string systems [2], and the subsequent development of the inverse scattering transform to integrate the Korteweg-de Vries equation [3,4] led to an intense search for exactly solvable systems described by dispersive nonlinear partial differential equations. More modern applications of such completely integrable systems include topological solitons in the context of magnetic skyrmions [5] and the Wess–Zumino–Witten model [6,7].

Among the mentioned class of exactly solvable systems is Toda's devise of a potential form that couples nearest neighbors in a lattice that allows for complete integration of the equation that governs the vibrations of the lattice structure [8]. A lattice with Toda's exponential interaction potential form admits the propagation of periodic waves and stable solitonic pulses. The Toda lattice equation has been shown to be formally equivalent to the equation describing the electric potential in a ladder circuit consisting of capacitors whose capacitance has a logarithmic dependence on the electric tension [9]. In its continuum limit, the one-dimensional Toda equation recovers the Korteweg-de Vries equation [10] while the two-dimensional Toda equation recovers [11] the Kadomtsev-Petviashvili equation, originally derived to model the effect of long transverse perturbations on the dynamics of plasma ion-acoustic modes of long wavelength and small amplitude [12].

For many nonlinear problems, it is useful to have their solutions expressed in terms of solutions to linear problems, as is the case of approaches that express the solutions of Liouville's equation in terms of solutions of Laplace's equation [13,14] and the case of the Cole-Hopf transform [15,16] that maps solutions of the Burgers' equation onto solutions of a linear diffusion equation [17]. Inspired by the former, this note develops a technique to express solutions, both solitonic and non-solitonic, of the two-dimensional Toda lattice equation (with two continuous variables X, Y and one discrete variable $Z \equiv n$) in terms of solutions of Laplace's equation, for which existence and uniqueness of solutions are guaranteed for given boundary conditions. The two-dimensional Toda lattice equation with its corresponding exponential restoring force reads [10,18]

$$\alpha u_{XX}(X,Y,n) + \beta u_{YY}(X,Y,n) = \\ = e^{u(X,Y,n-1)-u(X,Y,n)} - e^{u(X,Y,n)-u(X,Y,n+1)}. \quad (1)$$

To exploit the properties of Laplace's equation, the continuous variables will be rescaled as

$$x \equiv X/\sqrt{\alpha}, \quad y \equiv Y/\sqrt{\beta}, \quad (2)$$

hence

$$u_{xx}(x,y,n) + u_{yy}(x,y,n) = \\ = e^{u(x,y,n-1)-u(x,y,n)} - e^{u(x,y,n)-u(x,y,n+1)}. \quad (3)$$

Solutions to Eq. (3) will be sought using the ansatz

$$u(x,y,n) = F(\phi(x,y), n) + \log\left[|\nabla\phi(x,y)|^{-2n}\right], \quad (4)$$

where $\phi(x, y)$ is a non-trivial solution of the two-dimensional Laplace's equation

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https://doi.org/10.1016/j.physleta.2020.126979
0375-9601/

$$\phi_{xx} + \phi_{yy} = 0 \tag{5}$$

and $F(\phi, n)$ is a function to be determined. The present derivation takes advantage of the property noted by Clemente [14] that, if (5) holds, then also

$$\left[\log\left(|\nabla\phi|^{\gamma}\right)\right]_{xx} + \left[\log\left(|\nabla\phi|^{\gamma}\right)\right]_{yy} = 0 \tag{6}$$

is true $\forall\gamma \in \mathbb{C}$. In addition, the calculation will use the fact that, for any function $\psi \equiv \psi(\phi(x, y), n)$, the relation

$$\nabla^2\psi = \frac{\partial\psi}{\partial\phi}\nabla^2\phi + \frac{\partial^2\psi}{\partial\phi^2}|\nabla\phi|^2 \tag{7}$$

holds as long as the Laplacian and the gradient are two-dimensional in x, y and therefore independent of the variable n. Eq. (7) has been employed in tokamak plasma modeling, to formally construct solutions for the poloidal flux stream function ψ of a pressure-anisotropic axisymmetric equilibrium in terms of solutions of the isotropic Grad-Shafranov equation [19]. Such technique involves an integral transform and can also be used in the presence of plasma rotation [20] and for non-orthogonal coordinate systems [21].

Substituting the ansatz (4) into Eq. (3) and simplifying the resulting expression using Eqs. (5), (6) and (7), the dimensionality of the problem is reduced from (x, y, n) to (ϕ, n) and Eq. (3) becomes

$$\frac{\partial^2}{\partial\phi^2}F(\phi(x, y), n) = e^{F(\phi(x,y),n-1)-F(\phi(x,y),n)} - e^{F(\phi(x,y),n)-F(\phi(x,y),n+1)}. \tag{8}$$

Note that the ansatz (4) was chosen in such a way as to exactly cancel out the dependencies on $|\nabla\phi|$ that otherwise would be present in Eq. (8). The fact that formally Eq. (8) only depends on two independent variables, rather than three as is the case of Eqs. (1) and (3), allows for a simplified construction of conservation laws relative to Ref. [22]. Eq. (8) can be expressed in terms of the function $r(\phi, n) \equiv F(\phi, n) - F(\phi, n-1)$ as

$$\frac{\partial^2 r(\phi, n)}{\partial\phi^2} = 2e^{r(\phi,n)} - e^{r(\phi,n-1)} - e^{r(\phi,n+1)} \tag{9}$$

Defining $f(\phi, n)$ via

$$e^{-F(\phi,n)+F(\phi,n-1)} = e^{-r(\phi,n)} = c + \frac{\partial^2}{\partial\phi^2}\log f(\phi, n), \tag{10}$$

with c being a constant, allows Eq. (9) to be cast in Hirota's bilinear form [23], following a similar procedure as done in the one-dimensional case [10]. The substitution of Eq. (10) into Eq. (9) leads to

$$e^{-F(\phi,n)+F(\phi,n-1)} = \frac{f(\phi, n+1)\, f(\phi, n-1)}{f^2(\phi, n)} e^{c_1\phi+c_2} \tag{11}$$

where c_1 and c_2 are constants of integration. A solution can then be obtained through the ansatz

$$f(\phi, n) = 1 + e^{p\phi+\bar{\omega}n}, \tag{12}$$

which yields

$$e^{-F(\phi,n)+F(\phi,n-1)} = \left[1 + \frac{p^2}{4}\mathrm{sech}^2\left(\frac{p\phi+\bar{\omega}n}{2}\right)\right] e^{c_1\phi+c_2}, \tag{13}$$

where $\sinh(\bar{\omega}/2) = \pm p/2$. Or, in terms of the original function as defined by (4),

$$e^{-u(x,y,n)+u(x,y,n-1)} = \left[1 + \frac{p^2}{4}\mathrm{sech}^2\left(\frac{p\phi+\bar{\omega}n}{2}\right)\right] \frac{e^{c_1\phi+c_2}}{|\nabla\phi|^2}. \tag{14}$$

The one-soliton solution of Ref. [18] is exactly recovered with the particular choice of a traveling wave form for the Laplace's equation solution ϕ, linear in both x and y, specifically $\phi(x, y) = \left(\sqrt{\alpha}kx + \sqrt{\beta}ly\right)/p$ with $p = \pm\sqrt{\alpha k^2 + \beta l^2}$, $c_1 = c_2 = 0$. In that case $|\nabla\phi(x, y)|^2 = 1$. Using these choices and returning to the original variables using Eq. (2), Eq. (14) becomes

$$e^{-u(X,Y,n)+u(X,Y,n-1)} - 1 = \frac{\alpha k^2 + \beta l^2}{4}\mathrm{sech}^2\left[(kX + lY + \bar{\omega}n)/2\right]. \tag{15}$$

Eq. (15) is the same as Eq. 3.8 of Nakamura's paper [18]. It can be appreciated that the most stringent assumption of the present derivation was to assume a form for f (Eq. (12)). It should be noted, however, that other solutions, more convenient for a given physical problem at hand, can be constructed for different forms of f but still using the same methodology presented in this work, i.e., different f will lead to distinct $F(\phi, n)$ while still preserving the same solution ansatz (Eq. (4)). Because of its $e^{c_1\phi}$ ($c_1 \in \mathbb{C}$) dependence, Eq. (14) provides a direct means for constructing breather-like solutions [24], i.e., solutions with localization due to amplitude decay in the continuous variables and oscillation in the discrete variable or vice-versa, depending on the choice of the constants c_1, p and $\bar{\omega}$.

In summary, using the serendipitous identity (6) and an ansatz (4) that takes advantage of the structure [25] of the underlying equation (3), new solutions of the two-dimensional (nonlinear) Toda lattice equation were constructed in terms of solutions ϕ of the (linear) Laplace's equation. A particular choice of ϕ was shown to replicate a previously found solution for the lattice equation [18]. Other solutions can then be constructed from (14) to the limits of the imposed boundary conditions of a particular problem. For example, expression (14) provides a rapid means for construction of solutions with azimuthal symmetry. In that case, $(1/r)(r\phi_r)_r = 0$, where $r = \sqrt{X^2/\alpha + Y^2/\beta}$. Therefore any $\phi \propto \log r$ in Eq. (14) will lead to an azimuthally invariant solution to the two-dimensional Toda lattice.

CRediT authorship contribution statement

The manuscript has been authored by a single author.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

Discussions with V.L. Quito are appreciated. This work was supported by the US Department of Energy under contract DE-AC02-09CH11466.

References

[1] N.J. Zabusky, M.D. Kruskal, Phys. Rev. Lett. 15 (1965) 240.
[2] E. Fermi, P. Pasta, S. Ulam, M. Tsingou, Studies of Nonlinear Problems, Tech. Rep., Los Alamos Scientific Lab., NM, 1955.
[3] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Phys. Rev. Lett. 19 (1967) 1095.
[4] P.D. Lax, Commun. Pure Appl. Math. 21 (1968) 467.

[5] A.N. Bogdanov, C. Panagopoulos, Nat. Rev. Phys. 1 (2020).
[6] J. Wess, B. Zumino, Phys. Lett. B 37 (1971) 95.
[7] E. Witten, Nucl. Phys. B 223 (1983) 422.
[8] M. Toda, J. Phys. Soc. Jpn. 22 (1967) 431.
[9] R. Hirota, K. Suzuki, J. Phys. Soc. Jpn. 28 (1970) 1366.
[10] M. Toda, Theory of Nonlinear Lattices, vol. 20, Springer Science & Business Media, 2012.
[11] R. Hirota, J. Phys. Soc. Jpn. 50 (1981) 3785.
[12] B.B. Kadomtsev, V.I. Petviashvili, Sov. Phys. Dokl. 15 (1970) 539.
[13] G.W. Walker, Proc. R. Soc. Lond. A 91 (1915) 410.
[14] R.A. Clemente, Int. J. Math. Educ. Sci. Technol. 23 (1992) 620.
[15] J.D. Cole, Q. Appl. Math. 9 (1951) 225.
[16] E. Hopf, Commun. Pure Appl. Math. 3 (1950) 201.
[17] The transformation [26] that allows the Eckhaus equation to be cast in the form of a linear Schrodinger equation is another remarkable example.
[18] A. Nakamura, J. Phys. Soc. Jpn. 52 (1983) 380.
[19] R.A. Clemente, Nucl. Fusion 33 (1993) 963.
[20] R.A. Clemente, Plasma Phys. Control. Fusion 36 (1994) 707.
[21] L.C. Souza, R.L. Viana, Phys. Plasmas 26 (2019) 042502.
[22] K. Kajiwara, J. Satsuma, J. Math. Phys. 32 (1991) 506.
[23] R. Hirota, Prog. Theor. Phys. 52 (1974) 1498.
[24] N. Flytzanis, S. Pnevmatikos, M. Remoissenet, J. Phys. C 18 (1985) 4603.
[25] The method works in one or two continuous dimensions (for which $\nabla^2\phi = 0 \Rightarrow \nabla^2 \log|\nabla\phi| = 0$) but not necessarily in higher dimensions. In addition, the method is most useful for the case of an interaction potential of exponential form, which takes advantage of the logarithmic form of the ansatz (4).
[26] F. Calogero, S. De Lillo, Inverse Probl. 3 (1987) 633.