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ABSTRACT

A method is developed to analytically determine the resonance broadening function in quasilinear theory from first principles, due to either Krook or Fokker-Planck scattering collisions of marginally unstable plasma systems where discrete resonance instabilities are excited without any mode overlap. It is demonstrated that a quasilinear system that employs the calculated broadening functions reported here systematically recovers the growth rate and mode saturation levels for near-threshold plasmas previously calculated from nonlinear kinetic theory. The distribution function is also calculated, which enables precise determination of the characteristic collisional resonance width.

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The collisional broadening of resonance lines is a universal phenomenon in physics. For example, in atomic physics, collisions lead to abrupt changes in the phase and plane of vibration, thereby destroying phase coherence and leading to uncertainty in the associated photon energy. This leads to broadening of the atom emission/absorption profile.[1,2] In plasma physics, decoherence of the orbital motion of resonant particles allows the reduction of reversible equations of motion into a diffusive system of equations that governs the resonant particle dynamics without detailed tracking of the ballistic motion—as is the case in the widely used quasilinear (QL) formulations in Refs. 3–5. In spite of being an essential element of the structure of QL theory, the determination of the appropriate collisional broadening resonance function has not yet been formulated. In this Letter, we show how to calculate the collisional resonance function from first principles and show that its use implies that a QL plasma system automatically replicates the nonlinear growth rate and the wave saturation levels calculated from full kinetic theory near marginality.[6,7] Moreover, it is shown that a QL theory can be constructed for a single resonance, provided that it experiences enough background stochasticity. The results enable realistic reduced modeling of diffusive transport observed in fusion devices.[8]

We shall show how the results of previous works that focused on the dynamics of plasma systems just above the marginal state for instability[6,7] can be interpreted within the context of QL theory. Berk *et al.* developed a method that calculated the transition from the linearly unstable regime to the nonlinear stabilized regime.[6] In their investigation, a cubic nonlinear time delay equation was derived and applied to a wide variety of plasma systems (e.g., the bump-on-tail problem in Q-machine-like devices, alpha particle induced instability that is crucial in burning plasmas,[9] and prediction of the emergence of wave frequency chirping in tokamaks[10]). These studies showed that, with stochastic mechanisms present, such as collisions and background turbulence, quasi-steady solutions could be found. Based on these results, a heuristic QL method was developed[11] that replicated the results of these stationary solutions, both near and far from marginal stability. This model was an extension of the collisionless QL theory developed by Kaufman.[5] Berk[11] suggested intuitive rules, relying on an arbitrarily chosen shape, for creating a resonance function (i.e., an envelope function that weights the strength of the resonant interaction in the diffusion coefficient) that broadened the singular delta functions that appear in Kaufman's theory. The aim of the present work is to show that just above the marginal instability state, a systematic QL theory can be developed, where one obtains a resonance function that integrates to unity, as physically expected. Without further assumption, what then emerges is the shape of the resonance function and the mode saturation level, which replicates the results of the original kinetic calculations.[6,7] The predicted saturation level of the kinetic theory resulted from the derivation of a rather complex time-delayed integrodifferential equation, which turned out to be identical to the evolution equation previously derived for a shear flow fluid problem involving Rossby waves.[12] In contrast, the predictions of this new QL theory are derived from a simple set of equations, which yields a clear understanding of the physical processes that are taking place. The QL theory that is developed is applicable to complex, multidimensional systems. In particular, the new theory is being applied to whole-device modeling of multiple Alfvénic instabilities that are driven by energetic beams and fusion products in tokamaks.[13]

In systems where symmetries exist, such as in an axisymmetric tokamak (where the kinetic energy, the canonical toroidal momentum, and the magnetic moment are invariants of the unperturbed motion), it is convenient to employ canonically conjugated action J and angle ξ variables, where the angle is an ignorable variable in the absence of a perturbation. For the tokamak case, in the presence of low-frequency perturbations (compared to ion cyclotron frequency), the magnetic moment is approximately preserved and there is the emergence of a second invariant, which is a linear combination of unperturbed action variables, in such a way that the relevant resonant dynamics is one dimensional in the vicinity of an isolated resonance determined by the condition $\Omega = \omega + n\omega_\varphi - p\omega_\theta = 0$, where ω_θ and ω_φ are the mean poloidal and toroidal transit frequencies of the unperturbed orbit, p is an integer, and ω is the frequency of a mode with toroidal number n. From unperturbed Hamilton's equations, a convenient frequency-like variable, which is a function of the relevant action, can be defined as $\dot{\xi} = \partial H_0(J)/\partial J \equiv \Omega(J)$,[14,15] where H_0 is the unperturbed Hamiltonian and the relevant action angle is defined as $\xi \equiv n\varphi - p\theta$, where φ and θ are the toroidal and poloidal action angles, respectively.[16]

Resonant particles are described via a distribution function $f(\xi, \Omega; t)$. t is the time and $\Omega = 0$ determines the resonance condition. The kinetic equation for a single resonance is

$$\frac{\partial f}{\partial t} + \Omega\frac{\partial f}{\partial \xi} + Re\left(\omega_b^2 e^{i\xi}\right)\frac{\partial f}{\partial \Omega} = C[f], \tag{1}$$

where the form for the collisional operator $C[f]$ is taken as either $\nu_K(F_0 - f)$, which are the creation and annihilation terms of the Krook model,[17] or $\nu_{scatt}^3 \partial^2 (f - F_0)/\partial\Omega^2$, which is the diffusive scattering operator,[18] and ν_K and ν_{scatt} are the effective collision frequencies. The details on the derivation of the one-dimensional collisional operator in Ω space can be found in Ref. 19. F_0 is the distribution function in the absence of wave perturbations. ω_b is the nonlinear trapping (bounce) frequency at a given resonance, which is proportional to the square root of the mode amplitude. The distribution can be written in the form $f(\xi, \Omega, t) = f_0(\Omega, t) + \sum_{l=1}^{\infty}(f_l(\Omega, t)e^{il\xi} + c.c.)$. Then, for $l \neq 0$, the kinetic equation [Eq. (1)] for f_l satisfies

$$\frac{\partial f_l}{\partial t} + il\Omega f_l + \frac{1}{2}\left(\omega_b^2 f'_{l-1} + \omega_b^{2*} f'_{l+1}\right) = \left\{-\nu_K f_l, \nu_{scatt}^3 f''_l\right\}, \tag{2}$$

where the brackets on the right hand side denote the use of either Krook or scattering operators. The distribution function is solved in the limit $\epsilon \equiv \omega_b^2/\nu_{K,scatt}^2 \ll 1$, where $f(\xi, \Omega, t) = \sum_{r=0}^{\infty}\epsilon^r f^{(r)}(\xi, \Omega, t)$ and $f^{(r)}(\xi, \Omega, t) = f_0^{(r)}(\Omega, t) + \sum_{l=1}^{\infty}(f_l^{(r)}(\Omega, t)e^{il\xi} + c.c.)$. Then, formally choosing $\epsilon = 1$, as standardly taken in perturbation theory, we straightforwardly find the inequalities $|f_0^{\prime(0)}| \gg |f_1^{\prime(1)}| \gg |f_0^{\prime(2)}|, |f_2^{\prime(2)}| \gg |f_1^{\prime(3)}|, |f_3^{\prime(3)}| \ldots$. The lowest order component of the distribution is simply $f_0^{(0)} = F_0$. In the Fourier coefficients of the distribution f_b, the prime denotes the derivative with respect to Ω.

Sufficiently close to the linear instability threshold, with even moderate collisionality, $\nu_{K,scatt}/(\gamma_{L,0} - \gamma_d) \gg 1$ is satisfied ($\gamma_{L,0}$ is the mode linear growth rate at $t = 0$ and γ_d is the background damping rate). In this case, the detailed time history is not essential for the description of the system's dynamics.[20] Then, one can disregard the time derivative in (2) on the basis that $\partial f_l/\partial t \sim (\gamma_L - \gamma_d)f_l \ll \nu_{K,scatt} f_l$. Therefore, the principal time dependency contribution to f_l comes from $\omega_b(t)$ rather than from a delayed time integral over the particle distribution's time history.

Starting with the Krook case, to first order in ω_b^2/ν_K^2, Eq. (2) gives

$$f_1^{(1)} = -\frac{\omega_b^2 F'_0}{2(i\Omega + \nu_K)}. \tag{3}$$

Noting that the reality constraint implies $f_{-1}^{(1)} = f_1^{*(1)}$, to second order in ω_b^2/ν_K^2, Eq. (2) gives

$$\frac{\partial f_0^{(2)}}{\partial t} + \frac{1}{2}\left(\omega_b^2\left[f_1^{\prime(1)}\right]^* + \omega_b^{2*} f_1^{\prime(1)}\right) = -\nu_K f_0^{(2)}. \tag{4}$$

The angle-independent part of the distribution is $f(\Omega, t) \equiv \int_0^{2\pi} d\xi f(\xi, \Omega, t)/2\pi = f_0^{(0)}(\Omega, t) + f_0^{(2)}(\Omega, t) = F_0(\Omega) + f_0^{(2)}(\Omega, t)$. Noting that, by construction, $\partial F_0/\partial t = 0$ and $|F'_0| \gg |f_0^{\prime(2)}|$, one then obtains from Eqs. (3) and (4) that the relaxation of $f(\Omega, t)$ is governed by the diffusion equation

$$\frac{\partial f(\Omega, t)}{\partial t} - \frac{\pi}{2}\frac{\partial}{\partial \Omega}\left[|\omega_b^2|^2 \mathcal{R}(\Omega)\frac{\partial f(\Omega, t)}{\partial \Omega}\right] = C[f], \tag{5}$$

where, for the Krook case, $\mathcal{R}(\Omega)$ is

$$\mathcal{R}_K(\Omega) = \frac{1}{\pi\nu_K\left(1 + \Omega^2/\nu_K^2\right)}. \tag{6}$$

A somewhat similar procedure can be employed for the scattering case. To first order in ω_b^2/ν_{scatt}^2, Eq. (2) becomes $\nu_{scatt}^3 f_1^{\prime\prime(1)} - i\Omega f_1^{(1)} = \omega_b^2 F'_0/2$, which can be integrated, giving

$$f_1^{(1)} = -\frac{F'_0\omega_b^2(t)}{2\nu_{scatt}}\int_{-\infty}^{0} ds e^{i\frac{\Omega}{\nu_{scatt}}s} e^{s^3/3}. \tag{7}$$

Equation (7) is then iterated in (2) to second order in ω_b^2/ν_{scatt}^2. Again, using that $\partial F_0/\partial t = 0$ and $|F'_0| \gg |f_0^{\prime(2)}|$, it is readily found from Eq. (4), with the scattering operator instead of the Krook operator, that $f(\Omega, t) \equiv F_0(\Omega) + f_0^{(2)}(\Omega, t)$ for the scattering case also satisfies an equation of the form of Eq. (5), with

$$\mathcal{R}_{scatt}(\Omega) = \frac{1}{\pi\nu_{scatt}}\int_0^{\infty} ds\ \cos\left(\frac{\Omega s}{\nu_{scatt}}\right)e^{-s^3/3}. \tag{8}$$

The resonance functions (6) and (8) are plotted in Fig. 1(a). The property $\int_{-\infty}^{\infty}\mathcal{R}(\Omega)d\Omega = 1$, expected for functions that replace a delta function, is automatically satisfied by both forms of the resonance function. For a self-consistent description, the QL diffusion Eq. (5) must be solved simultaneously with the equation for amplitude evolution

$$\frac{d|\omega_b^2|^2}{dt} = 2\left(\gamma_L(t) - \gamma_d\right)|\omega_b^2|^2, \tag{9}$$

and for the growth rate[21]

$$\gamma_L(t) = \frac{\pi}{4}\int_{-\infty}^{\infty} d\Omega \mathcal{R}\frac{\partial f(\Omega, t)}{\partial \Omega}, \tag{10}$$

with its initial value being given by

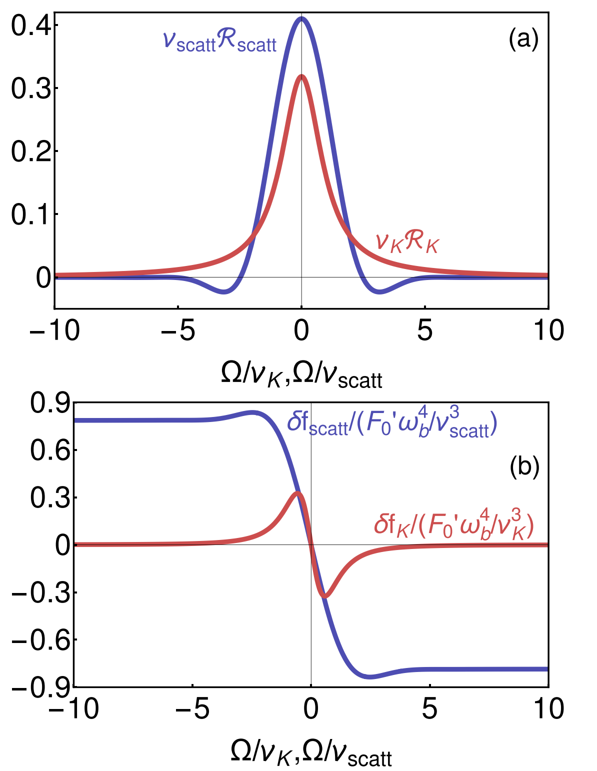

FIG. 1. (a) Resonance function [Eqs. (6) and (8)] and (b) $\delta f = f - F_0$ [Eqs. (12) and (13)] vs normalized frequency variable. The red and blue curves correspond to the Krook and scattering cases, respectively. The full width at half maximum of the resonance function in part (a) is $\Delta\Omega = 2\nu_K$ for Krook and $\Delta\Omega \cong 2.58\nu_{scatt}$ for the scattering case. The separation between the two peaks of each curve for δf in plot (b) is $\Delta\Omega = 2\nu_K/\sqrt{3}$ for Krook and $\Delta\Omega \cong 4.95\nu_{scatt}$ for the scattering case.

$$\gamma_{L,0} = \frac{\pi}{4}\int_{-\infty}^{\infty} d\Omega\mathcal{R}\frac{\partial F_0}{\partial\Omega} \approx \frac{\pi}{4}F_0'. \tag{11}$$

For marginally unstable kinetic theory, the saturation levels for Krook and scattering cases satisfy $|\omega_{b,sat}| \ll \nu_K, \nu_{scatt}$.[6,7] For near threshold QL theory, therefore, collisions are the main physical element broadening the resonances, and no contribution due to ω_b is present in Eqs. (6) and (8).

Interestingly, functions similar to (6) and (8) appear in the context of broadening of atomic emission lines: their equivalents are Eq. (12) of Ref. 22 and Eq. (5.68) (with $p=1$) of Ref. 23, respectively. Eq. (8) has the same form of the function calculated by Dupree[24] in a different context, namely, in the study of strong turbulence theory, where a dense spectrum of fluctuations diffuses particles away from their free-streaming trajectories (see Ref. 25 for a review covering broadening theories in strong turbulence). In that case, a diffusion equation with a constant coefficient is solved, a renormalized average propagator is introduced, and the cubic term in the argument of the exponential is proportional to a collisionless diffusion coefficient, unlike our results, where the shape of $\mathcal{R}$ is decoupled from the strength of the diffusion coefficient. In related approaches, as a result of collisions, the resonance singular layer broadens to an integral function, e.g., as has been found in Refs. 26 and 27.

A concern might arise about the physical significance of a resonance function that is negative in a part of its domain, as is shown in Fig. 1(a) for the function (8). We note that for the problem treated in the present work, the collisional diffusion ensures that the overall diffusion coefficient in Eq. (5) is always positive. In Dupree's study,[24] the assumed dense spectrum of overlapping turbulent modes ensures positivity over the entire phase-space domain.

To leading order near marginal instability, there emerges the following higher order steady state distribution functions [$\delta f \equiv f(\Omega,t) - F_0(\Omega)$] from Eq. (5). For the Krook model, it has the form

$$\delta f_K = -\frac{|\omega_b^2|^2}{\nu_K^3}\frac{\partial F_0}{\partial\Omega}\frac{\Omega/\nu_K}{\left(1+\Omega^2/\nu_K^2\right)^2}, \tag{12}$$

while for the diffusive scattering model,

$$\delta f_{scatt} = -\frac{|\omega_b^2|^2}{2\nu_{scatt}^3}\frac{\partial F_0}{\partial\Omega}\int_0^\infty \frac{ds}{s}\sin\left(\frac{\Omega s}{\nu_{scatt}}\right)e^{-s^3/3}. \tag{13}$$

Figure 1(b) shows the forms for the marginally unstable δf. These forms can be useful for code verification akin to studies reported in Ref. 28. Figure 1(b) is valid in the vicinity of the resonance, as long as F_0' can be assumed constant—its behavior far from the resonance would then be determined by the boundary conditions one imposes to Eq. (5).

We now demonstrate that near the instability threshold, the QL theory together with the calculated resonance functions [(6) and (8)] replicates the same saturation levels calculated by nonlinear theory.[6,7] Let us start with Eq. (5) for the Krook case. To leading order, it can be written as $-\frac{\pi}{2}|\omega_b^2|^2\frac{\partial F_0}{\partial\Omega}\frac{\partial\mathcal{R}_K}{\partial\Omega} = \nu_K(F_0 - f)$, since the marginality condition implies $\omega_b \ll \nu_K, \nu_{scatt}$. Differentiating with respect to Ω, then multiplying by $\mathcal{R}_K$ and integrating over Ω, we get

$$|\omega_b^2|^2\int_{-\infty}^{\infty}\mathcal{R}_K\frac{\partial^2\mathcal{R}_K}{\partial\Omega^2}d\Omega = \frac{-2\nu_K}{\pi\frac{\partial F_0}{\partial\Omega}}\int_{-\infty}^{\infty}\mathcal{R}_K\left(\frac{\partial F_0}{\partial\Omega} - \frac{\partial f}{\partial\Omega}\right)d\Omega. \tag{14}$$

Note that, because $\mathcal{R}$ vanishes at $\pm\infty$, integration by parts of the left hand side leads to $\int_{-\infty}^{\infty}\mathcal{R}_K\frac{\partial^2\mathcal{R}_K}{\partial\Omega^2}d\Omega = -\int_{-\infty}^{\infty}\left(\frac{\partial\mathcal{R}_K}{\partial\Omega}\right)^2 d\Omega = \frac{-1}{4\pi\nu_K^3}$ [the last equality follows from using the function given in Eq. (6)]. Recalling the definitions in Eqs. (10) and (11), it follows from Eq. (14) that $\gamma_L(t) = \gamma_{L,0}(1 - |\omega_b^2(t)|^2/8\nu_K^4)$. At saturation, i.e., when $\gamma_L = \gamma_d$, then $|\omega_{b,sat}| = 8^{1/4}(1-\gamma_d/\gamma_{L,0})^{1/4}\nu_K$, which is the same saturation level as the one predicted by the kinetic time-delayed integral nonlinear equation.[6]

A slightly different procedure can be employed for the scattering case, for which the QL diffusion Eq. (5) can be written to leading order as $-\frac{\pi}{2}|\omega_b^2|^2\frac{\partial F_0}{\partial\Omega}\frac{\partial\mathcal{R}_{scatt}}{\partial\Omega} = \nu_{scatt}^3\frac{\partial^2(f-F_0)}{\partial\Omega^2}$. Integrating over Ω, multiplying both sides by $\mathcal{R}_{scatt}$ and integrating over Ω, one obtains

$$|\omega_b^2|^2\int_{-\infty}^{\infty}\mathcal{R}_{scatt}^2 d\Omega = \frac{2\nu_{scatt}^3}{\pi\frac{\partial F_0}{\partial\Omega}}\int_{-\infty}^{\infty}\mathcal{R}_{scatt}\left(\frac{\partial F_0}{\partial\Omega} - \frac{\partial f}{\partial\Omega}\right)d\Omega. \tag{15}$$

The integration on the left hand side can be analytically performed using Eq. (8), which gives $\int_{-\infty}^{\infty}\mathcal{R}_{scatt}^2 d\Omega = \frac{2}{\pi\nu_{scatt}}\left[\Gamma\left(\frac{1}{3}\right)\left(\frac{3}{2}\right)^{1/3}\frac{1}{6}\right]^{-1}$. Using the definitions in Eqs. (10) and (11), then one obtains from

Eq. (15) that $\gamma_L(t) = \gamma_{L,0}[1 - |\omega_b^2(t)|^2 \Gamma(1/3)(3/2)^{1/3}/(6\nu_{scatt}^4)]$. At saturation, when $\gamma_L = \gamma_d$, then $|\omega_{b,sat}| \simeq 1.18(1-\gamma_d/\gamma_{L,0})^{1/4}\nu_{scatt}$, which is the same as what follows from nonlinear kinetic theory[7] QED.

The limit $\nu_{K,scatt}/(\gamma_{L,0} - \gamma_d) \gg 1$, when the detailed time history becomes unimportant, allows for the derivation of the analytical expression for the nonlinear growth rate $\gamma_{NL}(t) = \gamma_{L,0}(1 - \alpha|\omega_b^2(t)|^2)$,[20] where $\alpha = (8\nu_K^4)^{-1}$ for the Krook case and $\alpha = \Gamma(1/3)(3/2)^{1/3}/(6\nu_{scatt}^4)$ for the scattering case. Comparison with the above expressions for the calculated QL growth rates implies that they are equal to the nonlinear growth rate at all times for both collisional cases.

In conclusion, we have constructed a QL transport theory from first principles, which is able to account for the excitation of an isolated resonance. The conventional QL theory requires resonance overlapping to be obeyed, via the Chirikov criterion.[29] The present work shows that near marginal stability, the stochasticity introduced by collisions or background turbulence ensures that the particle orbital motion loses coherence in order to justify a diffusive approach. Besides, in the present work, the shape of the resonance function emerges naturally in the calculation, which, therefore, removes a major arbitrariness of the framework proposed in Ref. 11 and does so by means of a systematic derivation that does not require any assumption other than being close to marginal stability. In addition, it has been demonstrated that near marginal stability, the systematic QL theory we developed replicates the identical growth rates and saturation levels as predicted by a significantly more complex nonlinear kinetic theory based on solving a time delayed integrodifferential equation.[6,7] The demonstration did not rely on any assumption for the specific form of the distribution. Our demonstration assumed that the overall system is governed by a QL equation that self-consistently embodies collisional effects via a resonance function that had been previously determined from first principles [(6) and (8)].

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