RESISTIVE INSTABILITIES IN MAGNETIZED PLASMAS

Ricardo M. O. Galvão

Instituto de Física, Universidade de São Paulo
Laboratório de Física de Plasmas
C.P. 20516, CEP 01498 - São Paulo, SP, BRAZIL

Abstract

The theory of resistive instabilities in magnetized plasmas is presented at an introductory level, assuming that the reader is already familiar with ideal MHD modes. The linear and nonlinear behaviour of tearing modes and some relevant applications are discussed.

1. Introduction

The most violent instabilities in magnetically confined plasmas are the ideal magnetohydrodynamic (MHD) instabilities, which are driven unstable by pressure and current density gradients in ideally conducting fluids [1,2]. The condition of ideal conductivity, or zero resistivity, imposes a constraint on the allowed perturbed motions in the plasma. In fact, because the electric field in a frame moving with the fluid has to vanish, the magnetic flux through any surface moving with the fluid has to remain constant. This is referred to as the 'frozen-in law', which means that the magnetic fields is 'frozen' into the plasma, i.e., a fluid perturbation cannot slip with respect to the associated perturbed magnetic field. The frozen-in law and the ideal MHD instabilities are discussed in many references at introductory and advanced levels [1−5]. The global ideal MHD instabilities are not only violent, in the sense that they can destroy the equilibrium configuration, but also have a very large growth rate. For a typical cylindrical plasma column of minor radius a, the characteristic e-folding time is the poloidal Alfvèn time, i.e.,

$$\tau_{A\theta} = \frac{a}{V_{A\theta}}, \tag{1}$$

where $V_{A\theta} = B_\theta/\sqrt{\mu_0\rho}$ is the poloidal Alfvèn speed, B_θ is the poloidal component of the equilibrium magnetic field, and ρ is the fluid mass density [6]. For tokamaks this time is of the order of microseconds, too short for effective feedback control of the instabilities. Therefore, the parameters of actual confinement configurations have to be chosen such that the system remains stable to global ideal MHD modes.

It might be expected that dissipative mechanisms such as resistivity and heat conductivity have a beneficial effect on the stability of magnetic confinement configurations because they simply reduce the gradients of current density and pressure that drive the ideal MHD modes. However, it is important to realize that instead dissipation can produce new instabilities by removing constraints from the ideal fluid model and, thereby, making states of lower potential energy accessible to the system. In particular, the frozen-in constraint on the magnetic field is relaxed by taking into account the finite resistivity of the plasma. This has a profound effect on the MHD fluid model. With resistivity, the magnetic field tends to break up into a number of thin filaments, called *magnetic islands*, changing the topology of the equilibrium configuration. The resistive instability that leads to this break up is called the *tearing mode*. Usually the magnetic islands grow on a time scale much longer than the Alfvèn time scale; they reach a saturated size when the linear free energy available for driving the change in topology vanishes [7,8]. Experimental results indicate that the size of the saturated islands is not negligible in comparison with the radius of the plasma column [9,10]; they can therefore lead to enhanced transport because in the island region the heat flow across the field lines is essentially replaced by the much faster flow along the field lines [11].

In some cases, when the current density profile in a discharge shrinks due to some external effect, such as impurity influx, the resistive modes may become strongly unstable leading to large magnetic islands that touch each other or the material limiter inside the vacuum chamber. This mechanism has been proposed by many authors as the cause of the disruptive instability, a violent and fast event that disrupts the plasma column and induces large undesirable mechanical stresses on the vacuum chamber of tokamaks [12]. Finally, it has been suggested that the anomalous diffusion observed in tokamaks is due to the enhanced transport caused by a chain of small size magnetic islands (island width of the order of the ion gyroradius) that can be self-consistently sustained in the plasma column when kinetic effects are taken into account [13–16]. From this short introduction, it is evident that resistive modes and associated magnetic islands play an important role in the physics of magnetic confinement configurations. An equally relevant role is also played in astrophysical plasmas, where the mechanism of field line reconnection is extremely important [17].

The linear theory of resistive modes was originally discussed by Furth, Killeen, and Rosenbluth [18] and further developed by different authors [19–22]. A quasilinear model for the evolution of tearing modes was first proposed by Rutherford [7]. Many other non-linear models followed to explain the so-called sawtooth oscillations, major disruptions, and saturation of magnetic islands [8,23–26]. Although the theory of resistive instabilities is already discussed in different texbooks [27–30] and review papers [31–34], many students find the literature in this field quite hard to read. This is partly because the linear theory of tearing modes involve a singular perturbation problem for a at least fourth-order system of differential equations - even in the simplest geometry - and partly because of the apparently heuristic arguments and orderings that are used to extract the most relevant terms of the complicated equations. Actually, it is interesting to note that some people find the quasilinear Rutherford theory much more easy to apprehend than the linear theory of tearing modes. The objective of these lectures is not to review the entire subject, what has already been quite well done by other authors [17,31–33], but to present the main ideas and analytical techniques involved in the theory of resistive modes using a simple and systematic approach.

The characteristic time for the diffusion of the magnetic field caused by the finite resistivity of the plasma can be readily obtained from a dimensional analysis of the MHD fluid equations [4]. Considering the equations for the laws of Faraday, Ampere (without displacement current), and Ohm, i.e.,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{2}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j}, \tag{3}$$

and

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j}, \tag{4}$$

respectively, a equation for the evolution of the magnetic field can be derived by taking the curl of Eq.4 and substituting the result and Eq.3 into Eq.2. The evolution equation for the magnetic field becomes then

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{\eta}{\mu_0} \nabla^2 \vec{B}, \tag{5}$$

where η is the plasma resistivity and $\vec{v}$ is the fluid velocity (rationalized MKS units are used in these lectures). The first term in Eq.5 is a convective term whereas the second is a diffusive term. Let L and τ be respectively a characteristic length scale and a characteristic time scale for the physical phenomena that are being considered. Then, balancing $\partial \vec{B}/\partial t$ with the diffusive term, it follows that

$$\frac{B}{\tau_r} \sim \frac{\eta}{\mu_0} \frac{B}{L^2}, \tag{6}$$

and the characteristic time for resistive diffusion becomes

$$\tau_r = \frac{\mu_0 L^2}{\eta}. \tag{7}$$

It is well-known that the resistivity of fully ionized plasmas decreases with the 3/2 power of the electron temperature [30]. For high temperature plasmas the resistivity is quite small and one might suspect the dominant term in the right-hand-side of Eq.5 is the first and not the second one, which is proportional to η. To estimate the first term, another relation between $\vec{v}$ and $\vec{B}$ is necessary. This is provided by the momentum balance equation [4]

$$\rho(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla)\vec{v}=-\nabla p+\vec{j}\times\vec{B}, \tag{8}$$

where ρ and p are respectively the fluid mass density and scalar pressure. Using Ampere's law, Eq.3, this equation can be written in the form

$$\rho(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla)\vec{v}=-\nabla(p+\frac{B^2}{2\mu_0})+\frac{1}{\mu_0}(\vec{B}\cdot\nabla)\vec{B}. \tag{9}$$

Recalling that for magnetically confined plasmas $\beta=2\mu_0 p/B^2<1$ and using the same dimensional analysis, it follows from Eq.9 that

$$\rho\frac{v^2}{L}\sim\frac{B^2}{\mu_0 L} \qquad or \qquad v\sim V_A, \tag{10}$$

where the Alfvèn speed is given by $V_A=B/\sqrt{\mu_0\rho}$. Then the ratio between the second and first terms in Eq.5 scales as

$$\frac{(\eta/\mu_0)\nabla^2 B}{\nabla\times(\vec{v}\times\vec{B})}\sim\frac{B/\tau_r}{V_A(B/L)}=\frac{\tau_A}{\tau_r}, \tag{11}$$

where the Alfvèn characteristic time is defined as $\tau_A=L/V_A$. For high temperatures plasmas this ratio is a quite small number, i.e,

$$\epsilon\equiv\tau_A/\tau_r<10^{-5} \tag{12}$$

and it seems that the resistive diffusion term in Eq.5 is not relevant at all.

There is, however, a fallacy in the above argument. In the calculation of τ_A and τ_r, it has been assumed that the length scale is the same for both ideal ($\eta=0$) and resistive phenomena. Actually, if resistivity is neglected, the magnetic field is frozen into the plasma and the perturbed fluid motions can cause highly distorted field lines; as a consequence strong gradients of the perturbed magnetic field can appear in very narrow regions, of size much smaller than the global length scale L. In this case the resistive term in Eq.5 may become of the order of the convective term because the magnetic diffusion has to take place only in a very narrow layer, which will be called the *resistive layer* in the sequel. Let the size of the resistive layer be δL,where δ is a dimensionless quantity, $\delta\ll 1$. Substituting $\nabla^2 B\sim B/\delta^2 L^2$ into Eq.11, it follows that the resistive term becomes of the order of the convective term if $\delta\leq\epsilon^{1/2}$.

These arguments are the motivation for using the boundary layer technique [35] in the analytical theory of resistive modes. The relevant equations are solved in most of the plasma region neglecting resistivity; in the neighborhood of the singular points of the ideal equations, where strong variation of the perturbed quantities are expected, the effect of finite resistivity is taken into account. The width of this resistive layer and the mode growth rate are determined by asymptotically matching the solutions in the two regions[18]. A brief introduction to boundary layer theory will be presented in the section 3. However, the effect of resistivity on the gravitational interchange mode will be discussed first without using the formalism of boundary layer theory. This sequence of exposition is adopted in order to somewhat separate the physical problem from the mathematical technique used to analitically solve it.

2. Resistive Gravitational Mode in a Plasma Slab

The gravitational or Raleigh-Taylor instability is the classical fluid instability that occurs when a dense, incompresible fluid is supported against gravity by a less dense fluid [36]. In magnetically confined plasmas, the roles of the more dense and less dense fluids are played by the plasma and the confining magnetic field, respectively, and the role of gravity is played by the centrifugal acceleration felt by the particles flowing along the curved field lines. Indeed, in a curved field line, the centrifugal force on a particle guiding center is given by

$$\vec{F}_c = -mv_{\|}^2(\vec{b}\cdot\nabla)\vec{b}, \tag{13}$$

where $v_{\|}$ is the component of the velocity of the guiding center and $\vec{b}$ is the unit vetor parallel to the field lines [37]. To mock up the average curvature effect in a fluid model for the plasma, one can therefore assume a gravitational field given by

$$\vec{g} = -v_T^2(\vec{b}\cdot\nabla)\vec{b}, \tag{14}$$

where the average of $v_{\|}^2$ has been made equal to square of the thermal velocity v_T.

The plasma slab is the most simple model for a magnetically confined plasma. In this model, all fluid quantities such as density, pressure, and current density vary in only one direction, perpendicular to the plane of the confining magnetic field. The field-line curvature is mocked up by a constant gravitational field paralel to the equilibrium gradients, as indicated in Fig.1. Using cartesian coordinates, the equilibrium magnetic field is given by

$$\vec{B_0} = B_0\hat{e}_z + B_y(x)\hat{e}_y, \tag{15}$$

where the dependence of B_y on x represents the shear of the field lines. The fluid is supposed to be at rest so that equilibrium is provided by a balance between expansion and gravitational forces, i.e.,

$$\frac{d}{dx}(p + \frac{B_0^2}{2\mu_0}) = \rho g. \tag{16}$$

The stability analysis is carried out by finding out the normal modes of the system subject to small perturbations. Since the equilibrium quantities do not vary in the y and z directions, the perturbation can be assumed of the form

$$f(\vec{r},t) = f_0(x) + f_1(x)\exp[i(wt - \vec{k}\cdot\vec{r})], \tag{17}$$

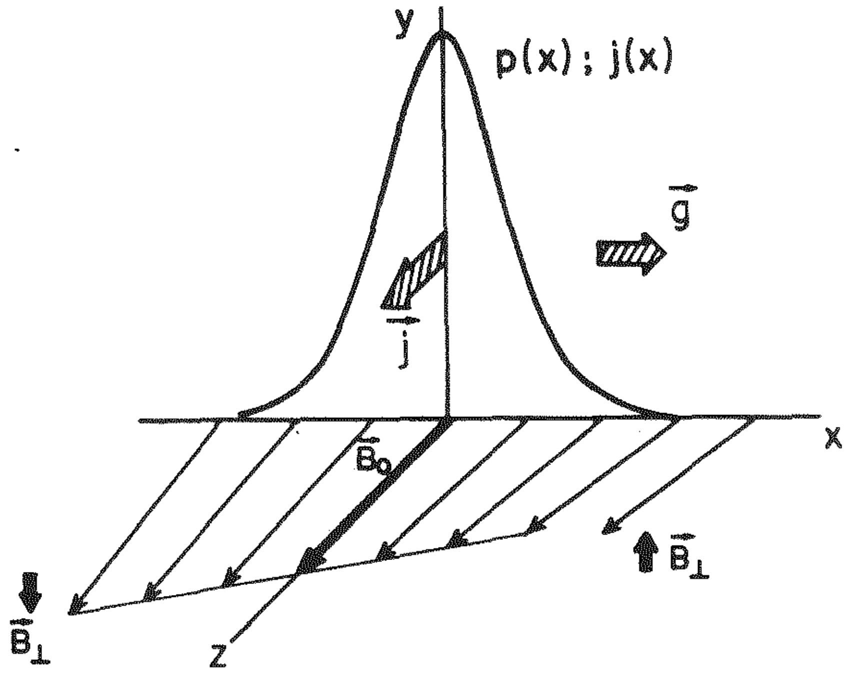

Figure 1: The plasma slab model.

where f represents any physical quantity, f_0 its equilibrium value, and f_1 the perturbation which is assumed small, i.e., $| f_1/f_0 | \ll 1$. The wavevector $\vec{k}$ is given by $\vec{k} = k_y\hat{e}_y + k_z\hat{e}_z$. If resistivity is neglected and only incompressible fluid displacements are considered such that $\nabla \cdot \vec{v}_1 = 0$, the equation for the perturbed magnetic field is readily obtained by linearizing Eq.5,

$$i\omega\vec{B}_1 = -v_{1x}\frac{dB_y}{dx}\hat{e}_y - i(\vec{k}\cdot\vec{B}_0)\vec{v}_1. \tag{18}$$

The component of the perturbed magnetic field perpendicular to the equilibrium field $\vec{B}_0$ is then given by $B_{1x} = -(\vec{k}\cdot\vec{B}_0)v_{1x}/\omega$. If the direction of $\vec{B}_0$ were constant, the direction of $\vec{k}$ could always be chosen such that $\vec{k}\cdot\vec{B}_0 = 0$ and no bending of the field lines would be produced by the perturbation. However, because of the finite shear, given by the x-dependence of B_y, the condition $\vec{k}\cdot\vec{B}_0 = 0$ can be satisfied only at some point in the plasma. In the neighborhood of this point B_{1x} does not vanish and the distortion of the field lines gives rise to a restoring force that opposes the instability driving force. This is the well-known stabilizing effect due to shear [4,5,28].

The magnitude of the restoring force is substantially reduced when the effect of non-vanishing plasma resistivity is taken into account, decoupling the plasma motion from the magnetic field. The dependence of the restoring force on plasma resistivity can be obtained from Ohm's law. Considering only the component of the equilibrium field parallel to $\vec{k}$, i.e., $B_{\|} = (k_yB_y + k_zB_z)\vec{k}/k^2$, and linearizing Eq.5, it follows that

$$\vec{E}_1 + \vec{v}_1 \times \vec{B}_{\|} = \eta\vec{j}_1. \tag{19}$$

A sufficient condition to decouple the fluid motion from the magnetic field is that no electric field is induced, that is, $\vec{E}_1 \simeq 0$, which means that the Ohmic dissipation is balanced by the motional electromotive force. In this case, Eq.19 can be simplified and the perturbed current density becomes

$$\vec{j}_1 \approx \frac{\vec{v_1} \times \vec{B}_{\|}}{\eta}. \tag{20}$$

The restoring force density is given by $\vec{F}_1 = \vec{j}_1 \times \vec{B}_{\|}$ and its component perpendicular to the equilibrium field is therefore

$$F_{1x} \simeq -\frac{v_{1x} B_{\|}^2}{\eta}. \tag{21}$$

For $\eta \to 0$, F_{1x}/v_{1x} is extremely large unless at the singular points where $B_{\|} = 0$. In the neighborhood of these points $B_{\|}$ can be approximated by $B_{\|} = (\delta a)(dB_{\|}/dx)$, where a is the width of plasma slab and δ is a small number such that δa gives the width of the resistive layer. The value of δ can be estimated by requiring that inside the resistive layer the restoring force be smaller than the perturbed driving force, $\vec{F}_{g1} = \rho_1 \vec{g}$. The perturbed density ρ_1 is obtained from the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0. \tag{22}$$

Linearizing this equation and recalling that $\nabla \cdot \vec{v}_1 = 0$, it follows that

$$\rho_1 = \frac{1}{\gamma} \frac{d\rho_0}{dx} v_{1x}, \tag{23}$$

and, therefore,

$$F_{g1} = \frac{1}{\gamma} v_{1x} g \frac{d\rho_0}{dx}, \tag{24}$$

where $\gamma = i\omega$ is the mode growth rate. Equating F_{g1} and F_{1x} and considering $B_{\|} = (\delta a)(dB_{\|}/dx)$, the expression for δ becomes

$$\delta = \left[\frac{\eta \mid g \frac{d\rho_0}{dx} \mid}{\gamma (a \frac{dB_{\|}}{dx})^2} \right]^{1/2}. \tag{25}$$

The mode growth rate can be estimated from the requirement that the work done by the perturbed driving force be equal to the increase in kinetic energy (the free magnetic energy is balanced out by Ohmic dissipation inside the resistive layer). The former is given by

$$\frac{dW}{dt} = v_{1x} F_{g1}, \tag{26}$$

whereas the latter is

$$\frac{dK}{dt} = \gamma\rho_0(v_{1x}^2 + v_{1\|\|})^2, \tag{27}$$

where $v_{1\|\|}$ is the component of the perturbed velocity parallel to the wave vector $\vec{k}$. From the incompressibility condition, $\nabla \cdot \vec{v}_1 = 0$, one has

$$\frac{dv_{1x}}{dx} + ikv_{1\|\|} = 0 \qquad or \qquad \frac{v_{1x}}{\delta a} \approx -ikv_{1\|\|}. \tag{28}$$

Therefore

$$\frac{dK}{dt} \approx \gamma\rho_0(1 + \frac{1}{\delta^2 k^2 a^2})v_{1x}^2 \approx \gamma\rho_0\frac{v_{1x}^2}{\delta^2 k^2 a^2}, \tag{29}$$

since $\delta \ll 1$. Equating dW/dt and dK/dt and using Eq.25, the expression for γ is readily obtained

$$\gamma \approx \left[\frac{\eta(ka)^2(g\frac{d\rho_0}{dx})^2}{\rho_0(a\frac{dB_{\|\|}}{dx})^2}\right]^{1/3}. \tag{30}$$

This expression shows that the mode growth rate is proportional to the 1/3 - power of the plasma resistivity or, what is equivalent, the characteristic time for the instability growth is much smaller than the resistive diffusion time accross the plasma slab. This can be better seen if the expression for γ is written in a more convinient form. Let the Alfvèn characteristic time be defined in terms of the parallel component of the magnetic field, i.e.,

$$\tau_A = \frac{a}{V_{A\|\|}} \qquad ; \qquad V_{A\|\|} = (a\frac{dB_{\|\|}}{dx})/\sqrt{\mu_0\rho}. \tag{31}$$

Then, rearranging the terms in Eq.30, the expression for γ can be written as

$$\gamma \approx (ka)^{2/3}(\frac{d\rho_0/dx}{\rho_0/a})^{2/3}(\tau_A^2\frac{g}{a})^{2/3}(\frac{\tau_A}{\tau_r})^{1/3}\tau_A^{-1}. \tag{32}$$

Thus the mode growth rate is proportional to $(\tau_A/\tau_r)^{1/3}\tau_A^{-1}$, that is, the instability grows slower than the ideal mode, which grows on the τ_A time scale [28], but much faster than the global resistive diffusion. This is a property of all resistive modes.

It is interesting to note that if the expression for $\vec{g}$ given in Eq.14 is used in Eq.32, one obtains $\tau_A^2 g/a \approx v_T^2/V_A^2 = \beta$, i.e., $\gamma \sim \beta^{2/3}$. The resistive interchange mode is therefore more relevant in high-β plasmas. Finally, it can be seen that this mode depends upon only the local conditions at the surface where $\vec{k} \cdot \vec{B} = 0$, which is called the *mode rational surface*. A more careful analysis of the eigenvalue equation indeed indicates that the mode eigenfunction is strongly localized at the mode rational surface [29]. For this reason, the resistive interchange mode is considered to be a local instability. Nevertheless, many continuous resistive interchange modes, each with its own mode rational surface and nearly equal growth rate, can couple together to form elongated convection cells that substantially increase the transport of energy across the field lines. This mechanism may become particularly relevant in high-β tokamak plasmas [38].

3. Intermezzo Matematico: Boundary Layer Technique

Although the plasma resistivity is usually quite small, it has an interesting property from a mathematical point of view. The order of the system of differential equations that results from linearizing the MHD equations is increased when resistivity is taken into account. If a solution of the linearized equations is sought as a series in powers of some small parameter proportional to resistivity, such as $\epsilon = \tau_A/\tau_r$, the lowest-order solution will come from a system of equations of lower order than original. Therefore, a boundary condition may be lost and the effect of non-vanishing resistivity, which is most important precisely where the solution has strong variations, is not properly described. This situation can be illustrated by the following simple example. Consider the differential equation

$$\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0, \tag{33}$$

where ϵ is a small dimensionless parameter, $\epsilon \ll 1$, and the boundary conditions

$$y(0) = 0 \qquad ; \qquad y(1) = 1. \tag{34}$$

If a series solution in powers of ϵ is sought, i.e.,

$$y(x) = y_o(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + ..., \tag{35}$$

the lowest-order equation becomes

$$\frac{dy_0}{dx} + xy_0 = 0, \tag{36}$$

and its solution is given by

$$y_o(x) = Ce^{-x^2/2}. \tag{37}$$

The constant C can be found by imposing the right boundary condition, $y_o(1) = 1$; it then follows that $C = exp(0.5)$ and the lowest-order solution becomes

$$y_0(x) = e^{(1-x^2)/2}. \tag{38}$$

It is clear however, that this solution *cannot* satisfy the left boundary condition, $y(0) = 0$, i.e., one boundary condition is lost when the limit $\epsilon \to 0$ is taken in Eq.33. Nevertheless, a solution of the original equation that satisfies both boundary conditions is certainly possible. If no approximations are made, Eq.33 can be solved analytically in terms of Bessel functions, or, what turns out to be simpler, integrated

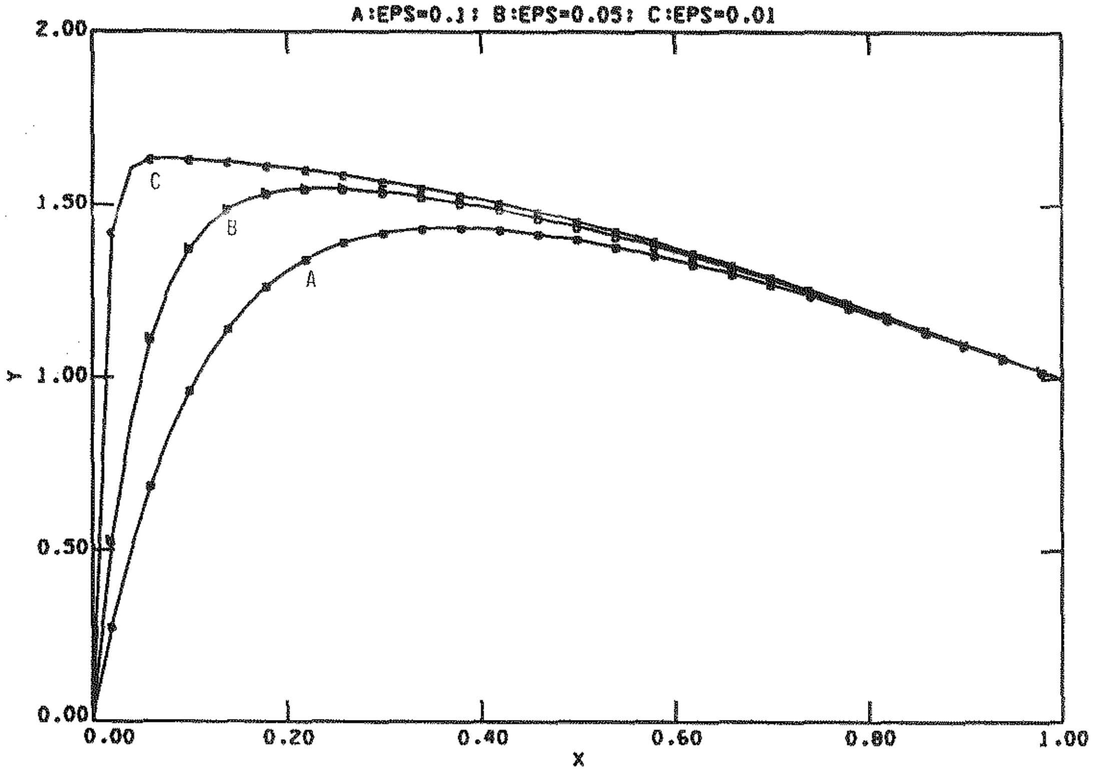

Figure 2: Plot of the solution of Eq.33, for different values of ϵ.

numerically. A plot of the numerical solution is shown in Fig.2 for different values of ϵ.

From this figure it can be seen that the power series solution fails because the function varies quite strongly within a distance of order ϵ from the left boundary. In this region, the second derivative becomes quite large, increasing as ϵ decreases, and the first term in Eq.33 becomes of the order of the others. This type of problem is called a *singular perturbation problem* and the region where the solution varies strongly is called a *boundary layer* [35]. The term boundary layer comes from Fluid Mechanics; it is sheath that is formed at the stationary boundaries of a viscous fluid flowing inside a pipe. The effect of the viscous term in the Navier-Stokes equation becomes important inside the boundary layer because of the large velocity gradients [39].

The basic ideia of the boundary layer technique to solve singular perturbation problems is to devide the solution domain into two regions. An 'outside' region, away from the point where the singular nature of the solution is manifested, and a region inside the boundary layer. In the outside region the small term that depends on ϵ is neglected and a solution of the resulting equation satisfying the proper boundary condition is looked for. Inside the boundary layer the small term is kept and the independent variable is re-scaled in order to make the small term of the same order of others in the equation. Because of the re-scaling process, some terms in the original equation may become negligible inside the boundary layer. The 'internal' solution is found and matched asymptotically to the 'outside' solution. By asymptotically matching it is meant that the outside solution in the limit $x \to 0$ has to match the inside solution in the limit $x_i \to \infty$, where x is the external independent variable and x_i is the re-scaled independent variable inside the boundary layer. In this procedure, the width of the boundary layer does not have to be imposed a priori, but it comes out naturally from the entire solution of the problem.

As an example, the boundary layer technique will be now applied to find an approximate solution to Eqs.33 and 34. In the outside region $\epsilon = 0$ and the proper solution is given by Eq.38. Inside the boundary layer the term proportional to ϵ has to be kept. Since usually the dependence of the width of the boundary layer on ϵ is not known a priori, the independent variable is re-scaled assuming that

$$x_i = \epsilon^{\alpha} x, \tag{39}$$

where α is a constant to be determined and $x_i = O(1)$. Carrying out this change of variables, Eq.33 becomes

$$\frac{d^2y}{dx_i^2} + \epsilon^{-(1+\alpha)}\frac{dy}{dx_i} + \epsilon^{-(1+3\alpha)}x_i y = 0. \tag{40}$$

The constant α is determined by imposing that highest derivative in the equation be balanced by at least some other term. If the first and third terms in Eq.40 are balanced, it follows that $\alpha = -1/3$. In this case, however, the second term becomes of order $\epsilon^{-2/3} \gg 1$, much larger than the terms that were kept. Balancing the first and second terms, on the other hand, yields $\alpha = -1$ and the third term becomes of order $\epsilon^2 \ll 1$ and can be neglected. Therefore this is the proper choice; the resulting equation becomes

$$\frac{d^2y}{dx_i^2} + \frac{dy}{dx_i} = 0, \tag{41}$$

and the corresponding solution which satisfies the boundary conditions $y(0) = 0$ is given by

$$y(x_i) = A(1 - e^{-x_i}), \tag{42}$$

The constant A is determined by the asymptotic matching procedure. The limit $x_i \to \infty$ yields $y \approx A$ whereas as $x \to 0$ the outside solution, Eq.38, gives $y \approx e^{\frac{1}{2}}$. Therefore $A = e^{\frac{1}{2}}$, and the internal solution, written in terms of the original variable, becomes

$$y(x) = e^{1/2}[1 - e^{-(x/\epsilon)}]. \tag{43}$$

The approximate solution for $\epsilon = 0.01$, calculated from Eqs.38 and 43, is shown in Fig.3 together with the exact (numerical) solution. The agreement between the two curves is quite good and improves as the value of ϵ decreases. For larger values of ϵ

the agreement is not satisfactory. To improve the approximation in this case, terms of higher order in ϵ have to be kept in the outside and inside solutions [35]. Fortunately, for resistive modes it is sufficient to keep only the lowest order terms.

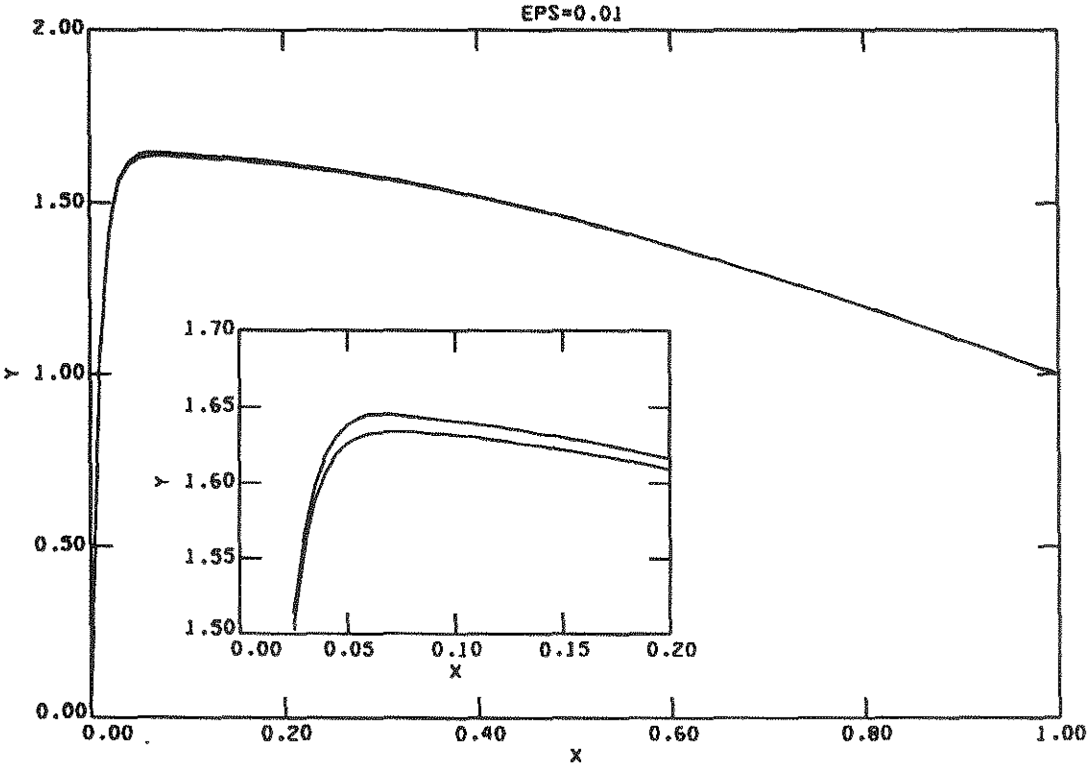

Figure 3: Boundary-layer (upper curve) and numerical (lower curve) solutions of Eq.33 and 34 for $\epsilon = 0.01$. The insert shows a blow up of the region close to the edge of the boundary layer .

4. Tearing Mode in a Plasma Slab

The resistive gravitational mode in a plasma slab was discussed in section 2 mainly to provide an introduction to the physical processes involved in the dynamics of resistive modes. The mode growth rate was estimated without solving the mode eigenvalue equation. This was possible because, as already mentioned, the resistive gravitational mode is very localized and its growth rate depends only on the local values of the equilibrium gradients at the mode rational surface. There is another resistive mode that can become unstable in a plasma slab and which is not localized. This mode is driven by the gradient of the current density, $j(x)$, and tears the opposing field lines of the field generated by the plasma current, $\vec{B}_1 = B_y(x)\hat{e}_y$, at $x = 0$, reconnecting them to form thin current filaments, called magnetic islands. This is the so-called *tearing mode* which plays a fundamental role in the physics of magnetically confined current-carrying plasmas [14,15,23,28,33,40]. The analytical theory of linear tearing modes is based upon the boundary layer technique discussed in the last section. Naturally, it is also possible to study tearing modes without approximations using numerical techniques [41]. However, the analytical approach is not only enlightening but also very useful because it allows a quick qualitative analysis of experimental results [33,40] and a smooth transition to the non-linear evolution of tearing modes [7].

The curvature of the field lines does not strongly affect the stability of global modes, such as the tearing mode [6], and therefore the gravitational field introduced in section 2 will be neglected here. The equilibrium equation is then given by Eq.16 with $g = 0$ and the equilibrium field is given by Eq.15. The perpendicular field, $\vec{B}_\perp = B_y(x)\hat{e}_y$, is produced by the plasma current flowing in the z direction and the longitudinal field, $\vec{B}_0 = B_0\hat{e}_z$, is supposed to be produced by external currents. To mock up the experimental situation in confinement configurations such as tokamaks, it is usually assumed that $\mid \vec{B}_\perp \mid /B_0 \ll 1$.

4.1. *Basic Equations*

It is convinient to introduce a flux function $\psi(x, y; t)$ to describe the total magnetic field (equilibrium plus perturbation) through the equation

$$\vec{B} = \nabla\psi \times \hat{e}_z + B_0\hat{e}_z, \tag{44}$$

such that the equation $\nabla \cdot \vec{B} = 0$ is automatically satisfied. To simplify the analysis, a two-dimensional configuration is assumed such that all physical quantities depend only on the (x,y) coordinates. Substituing Eq.44 into Ampere's law, Eq.3, yields an equation for ψ

$$\nabla^2\psi = -\mu_0 j, \tag{45}$$

where j is the current density along the z axis. Assuming only incompressible fluid motions, $\nabla \cdot \vec{v} = 0$, the fluid velocity can also be obtained from a flux function, $\phi(x, y; t)$, i.e

$$\vec{v} = \nabla\phi \times \hat{e}_z, \tag{46}$$

The flux functions ψ and ϕ will be the basic quantities used to describe the linear and nonlinear dynamics of tearing modes. In the sequel they will be referred to as the magnetic flux function and the vorticity, respectively.

The equation for the time evolution of the magnetic flux function ψ can be obtained from the laws of Faraday and Ohm. Calculating the curl of Eq.4, substituting into Eq.2, and using Eq.44 it follows that

$$\nabla \times [\eta\vec{j} - \vec{v} \times (\nabla\psi \times \hat{e}_z + B_0\hat{e}_z) + \frac{\partial\psi}{\partial t}\hat{e}_z] = 0. \tag{47}$$

Therefore the quantity inside the square brackets has to be equal to the gradient of a scalar function, which is the electrostatic potencial ϕ_e. Taking the z - component of this quantity yields

$$\frac{\partial \psi}{\partial t} + \vec{v} \cdot \nabla \psi = -\eta j + E_0, \tag{48}$$

where $E_0 = -\partial \phi_e / \partial z$ is the electric field in the direction of $\vec{B}_0$ which is necessary to maintain the resistive equilibrium.

The evolution equation for the vorticity ϕ is obtained from the equation of motion, Eq.8, but it requires a somewhat more cumbersome algebra. Taking the curl of Eq.8, one obtains $\nabla \times \nabla p = 0$ and two more complicated terms remain, $\nabla \times [\rho(\vec{v} \cdot \nabla)\vec{v}]$ and $\nabla \times (\vec{j} \times \vec{B})$. The first term can be calculated using the vector identity

$$\nabla \times [\rho(\vec{v} \cdot \nabla)\vec{v}) = \rho(\vec{v} \cdot \nabla)(\nabla \times \vec{v}) + [\nabla \rho(\vec{v} \cdot \nabla)] \times \vec{v}. \tag{49}$$

Then, assuming ρ =const and using the incompressibility constraint, $\nabla \cdot \vec{v} = 0$, and Eq.46, it follows that

$$\nabla \times \vec{v} = -\hat{e}_z \nabla^2 \phi \qquad \textit{and} \qquad [\nabla \rho(\vec{v} \cdot \nabla)] \times \vec{v} = 0. \tag{50}$$

Therefore

$$\nabla \times [\rho(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla)\vec{v}] = -\rho(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla)(\hat{e}_z \nabla^2 \phi). \tag{51}$$

The term $\nabla \times (\vec{j} \times \vec{B})$ can be readily calculated using Eq.44 and simple vector identities; the result is

$$\nabla \times (\vec{j} \times B) = -[(\hat{e}_z \cdot \nabla \psi)\nabla j)] \times \hat{e}_z - [j(\hat{e}_z \cdot \nabla)\nabla \psi] \times \hat{e}_z = \nabla j \times \nabla \psi. \tag{52}$$

Equating the z - components of Eqs.51 and 52 yields the evolution equation for the vorticity,

$$\rho(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla)\nabla^2 \phi = \hat{e}_z \cdot (\nabla \psi \times \nabla j). \tag{53}$$

The two basic evolution equations are then Eqs.48 and 53, with the current density given by Eq.45. They form a system of coupled non-linear equations for ψ and ϕ, with the non-linearities coming from the convective derivative operator, $\vec{v} \cdot \nabla$, and the Lorentz force term, $\nabla \psi \times \nabla j$.

To solve analytically Eqs.48 and 53, even in their linearized versions to be presented later, it is necessary to identify a small parameter. From the discussions in the introduction and in Section 2, it is clear that this small parameter should be the ratio of the Alfvèn characteristic time to the resistive diffusion time, $\epsilon = \tau_A / \tau_r$. This parameter can be brought out in Eqs.48 and 53 by a suitable normalization of the physical quantities. Defining the perpendicular Alfvèn velocity

$$V_A = \frac{B_{\perp 0}}{\sqrt{\mu_0 \rho}}, \tag{54}$$

where $B_{\perp 0}$ is the constant value of the equilibrium perpendicular field for away from the current layer (see Fig.1), the Alfvèn characteristic time is given by $\tau_A = a/V_A$. Time is then normalized to τ_A, velocity to V_A, magnetic flux function to $aB_{\perp 0}$, vorticity to aV_A, and electric field to $B_{\perp 0}V_A$. Using Eq.45, the normalized equations become

$$\frac{\partial \psi}{\partial t} + \vec{v} \cdot \nabla \psi = \epsilon \nabla^2 \psi + E \tag{55}$$

and

$$(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla) \nabla^2 \phi = -\hat{e}_z \cdot [\nabla \psi \times \nabla(\nabla^2 \psi)], \tag{56}$$

where the small parameter ϵ appears explicitly (notice the gradient operators are also normalized to a^{-1} in the above equations). These equations will be used to describe the equilibrium, linear stability, and non-linear evolution of the tearing modes.

4.2. *Linear Analysis*

The linear growth rate of the tearing modes is determined from the solution of the equations that result from the linearization of Eqs.55 and 56, subjet to proper boundary conditions. The linearization is carried out by assuming that

$$\psi(x, y; t) = \psi_0(x) + \psi_1(x, y)e^{\gamma t} \tag{57}$$

and

$$\phi(x, y; t) = \phi_1(x, y)e^{\gamma t}, \tag{58}$$

where γ is the mode growth rate, ψ_o is the magnetic flux function describing the equilibrium field, and $\mid \psi_1 \mid / \mid \psi_o \mid \ll 1$. A stationary equilibrium has been assumed such that $\phi_o = 0$. The equilibrium magnetic field is given by $\vec{B}_\perp = -(d\psi_0/dx)\hat{e}_y$ and the equilibrium current is maintained by the electric field, i.e., $\epsilon \nabla^2 \psi_0 = -E$. To facilitate the notation in the following, let $f(x) = -d\psi_0/dx$. Neglecting second-order terms, the linearized equations become

$$\gamma \psi_1 + \vec{v}_1 \cdot \nabla \psi_0 = \epsilon \nabla^2 \psi_1 \tag{59}$$

and

$$\gamma \nabla^2 \phi_1 = -\hat{e}_z \cdot [\nabla \psi_0 \times \nabla(\nabla^2 \psi_1) + \nabla \psi_1 \times \nabla(\nabla^2 \psi_0)], \tag{60}$$

where $\nabla\psi_0 = -f(x)\hat{e}_z$ and $\vec{v}_1 \cdot \nabla\psi_0 = -f(x)\partial\phi/\partial y$. Since the equilibrium quantities depend only on x, the perturbed quantities can be Fourier analysed in the y coordinate. A careful inspection of Eq.60 shows that ψ_1 and ϕ_1 have opposite parities; therefore the Fourier decomposition is chosen in the form

$$\psi_1(x,y) = \psi_1(x)\cos(ky) \tag{61}$$

and

$$\phi_1(x_1 y) = \phi_1(x)\sin(ky) \tag{62}$$

where the wavenumber k is normalized to a^{-1}. Substituting these expressions into Eqs.59 and 60 and carrying out the vector operations, the perturbed equations become

$$\psi_1 = \frac{k}{\gamma} f\phi_1 + \frac{\epsilon}{\gamma}[\psi_1'' - k^2\psi_1] \tag{63}$$

and

$$\phi_1'' - k^2\phi_1 = -\frac{k}{\gamma} f[\psi_1'' - k^2\psi_1] + \frac{k}{\gamma} f''\psi_1, \tag{64}$$

where prime means derivative with respect to x.

The system of equations above has to be solved with the proper boundary conditions at the edge of the plasma slab ($x = \pm a$) or at $x \to \pm\infty$, if the equilibrium quantities are assumed to vanish away monotonically. It will be assumed that $\gamma \to 0$ as $\epsilon \to 0$, that is, that the tearing mode exists only in the presence of non-vanishing resistivity. This is true for the tearing mode but not for the resistive internal kink mode, which can be unstable already in the ideal limit [21]. The internal kink mode occurs only in cylindrical or toroidal geometry and it will not be considered in this section. If the resistive parameter is made to vanish, $\epsilon \to 0$, it results a simple second order equation for ψ_1, whereas the original system of equations is of fourth order. This indicates that ϵ introduces a singular perturbation, as discussed in section 3.

The character of the singular perturbation and the existence of the boundary layer can be better seen by analysing the equation that results from taking the limit $\epsilon \to 0$ in Eqs.63 and 64. From the former one obtains $\phi_1 = (\gamma/kf)\psi_1$ and the latter becomes

$$\psi_1'' - (k^2 + \frac{f''}{f})\psi_1 = -\frac{\gamma^2}{k^2}[(\frac{\psi_1}{f})'' - k^2\frac{\psi_1}{f}]. \tag{65}$$

Since γ is expected to become very small in the limit $\epsilon \to 0$, this equation can be approximated as

$$\frac{d^2\psi_1}{dx^2} - (k^2 + \frac{1}{f}\frac{d^2 f}{dx^2})\psi_1 = 0 \tag{66}$$

where the x derivative has been written out explicitly. The solution of this equation depends on the function $f(x)$. Since the perpendicular field $\vec{B}_\perp = f(x)\hat{e}_y$ vanishes at

$x = 0$, so does the function f. Therefore, Eq.66 has a singular point at $x = 0$ and in its neighborhood the solution $\psi_1(x)$ is expected to vary strongly. This indicates that a boundary layer has to be considered in the neighborhood of the origin. The character of the singular perturbation problem in this case is somewhat different from the prototype example discussed in section 3. In this case it is not a boundary condition that cannot be satisfied by the lowest-order equation but, because of the singularity at the origin, the solution of Eq.66 diverges at $x = 0$ and the two pieces for $x < 0$ and $x > 0$ are discontinuous. Introducing a boundary layer at the origin allows the continuity of the solution and the determination of all relevant constants. Then, following the scheme described in section 3, two regions are considered: a resistive (or boundary) layer close to the origin and an ideal region away from the origin. In the ideal region, Eq.66 has to be solved whereas in the resistive layer the original equations, Eqs.63 and 64, have to be solved using a re-scaled independent variable.

4.2.1. *Solution in the ideal region*

Unless for very simple equilibrium models, the solution of Eq.66 has to be numerically obtained. However, the asymptotic behavior of the solution as $x \to 0$ can be readily found using the method of Frobenius. At the origin $d\psi_0/dx = 0$ and the equilibrium magnetic flux function can be expanded in a Taylor series. Then Eq.66 becomes

$$\frac{d^2\psi_1}{dx^2} - \frac{K}{x}\psi_1 = 0, \tag{67}$$

where

$$K \equiv \frac{(d^3\psi_0/dx^3)_0}{(d^2\psi_0/dx^2)_0}. \tag{68}$$

The solution of Eq.67 is given by [42]

$$\psi_1(x) = A\psi_r(x) + B\psi_s(x), \tag{69}$$

where $\psi_r(x)$ is the regular solution,

$$\psi_r(x) \approx Kx + \frac{1}{2}K^2x^2 + \frac{1}{12}K^3x^3 + ... \tag{70}$$

and $\psi_s(x)$ is the irregular solution,

$$\psi_s(x) \approx 1 - Kx - \frac{5}{4}K^2x^2 + ... + \psi_r(x)\ln \mid x \mid . \tag{71}$$

Therefore, as $x \to 0$, the dominant behavior of the solution in given by

$$\psi_1(x) = C_1 + C_2 x \ln \mid x \mid \tag{72}$$

This equation shows that $\psi_1(x)$ is actually constant but its first and higher derivatives diverge at $x = 0$. This asymptotic behavior is the origin of the so-called 'constant

- ψ approximation', which looks artificial to beginners in the subject. Considering the form of the asymptotic solution in the ideal region, it is clear that there can be a jump in the logarithmic derivative of $\psi_1(x)$ across the resistive layer; this jump is given by

$$\Delta' = \frac{1}{\psi_1(0)}\left[(\frac{d\psi_1}{dx})_+ - (\frac{d\psi_1}{dx})_-\right]. \tag{73}$$

The asymptotic matching to the solution inside the resistive layer can be carried out by simply requiring the Δ' be the same for the inner and outer solutions.

4.2.2. *Solution inside the resistive layer*

The asymptotic solution in the ideal region, Eq.72, suggests that the perturbed magnetic flux function inside the resistive is, to lowest order, a constant. The scaling of the non-constant part of ψ_1 and of ϕ_1 with ϵ is, however, not known a priori. Therefore, generalizing the boundary-layer thecnique discussed in section 3, not only the independent but also the dependent variables and the growth rate γ are re-scaled as follows.

$$x = \epsilon^b \bar{x}, \tag{74}$$

$$\psi_1(x) = \bar{\psi}_{10} + \epsilon^b \bar{\psi}_{11}(\bar{x}), \tag{75}$$

$$\phi_1(x) = \frac{\gamma}{k}\epsilon^c \bar{\phi}_1(\bar{x}), \tag{76}$$

and

$$\gamma = \epsilon^a \Lambda. \tag{78}$$

where the bar denotes the re-scaled variables, i.e., $\bar{x} \sim \bar{\psi}_{11} \sim \bar{\phi}_1 \sim O(1)$, $\bar{\psi}_{10}$ and Λ are constants also of order one, and a,b, and c are constants to be determined. The factor (γ/k) was included in Eq.76 for later convinience. Since it has been assumed that the normalized perpendicular field vanishes linearly with x at the origin, inside the resistive layer the function $f(x)$ is given by $f(x) \approx x = \epsilon^b \bar{x}$, and $f''(x)$ can be neglected. Substituting Eqs.74 to 78 into Eqs.63 and 64, results in the two following equations for the re-scaled variables

$$\bar{\psi}_{10} + \epsilon^b \bar{\psi}_{11} = \epsilon^{b+c}\bar{x}\bar{\phi}_1 + \frac{\epsilon^{1-a-b}}{\Lambda}[\frac{d^2\bar{\psi}_{11}}{d\bar{x}^2} - k^2\epsilon^b(\bar{\psi}_{10} + \epsilon^b\bar{\psi}_{11})] \tag{79}$$

and

$$\frac{d^2\bar{\phi}_1}{d\bar{x}^2} - k^2\epsilon^{2b}\bar{\phi}_1 = -\frac{k^2}{\Lambda^2}\epsilon^{2b-2a-c}\bar{x}[\frac{d^2\bar{\psi}_{11}}{d\bar{x}^2} - k^2\epsilon^b(\bar{\psi}_{10} + \epsilon^b\bar{\psi}_{11})]. \tag{80}$$

The constants a, b, and c have to be chosen such that the second derivative terms which are proportional to some power of ϵ become of order one together with the other relevant terms in Eqs.79 and 80. This is achieved by imposing that $b+c=0$ and $1-a-b=0$ in Eq.79 and $2b-2a-c=0$ in Eq.80; therefore

$$a=\frac{3}{5} \quad ; \qquad b=\frac{2}{5} \quad ; \qquad c=-\frac{2}{5}. \tag{81}$$

Substituting these constants into eqs.79 and 80 and neglecting terms of higher order in ϵ, one obtains

$$\frac{d^2\bar{\psi}_{11}}{d\bar{x}^2}=-\frac{\Lambda^2}{k^2}\frac{1}{\bar{x}}\frac{d^2\bar{\phi}_1}{d\bar{x}^2} \tag{82}$$

and

$$\frac{d^2\bar{\phi}_1}{d\bar{x}^2}-\frac{k^2}{\Lambda}\bar{x}^2\bar{\phi}_1=-\frac{k^2}{\Lambda}\bar{\psi}_{10}\bar{x}. \tag{83}$$

The constant Λ, which determines the mode growth rate through Eq.78, is actually an eigenvalue which will be determined from the matching with the outside solution, as discussed in the next sub-section. To solve Eq.83, it is convenient to introduce the following variables

$$\chi=-\frac{(\Lambda/k^2)^{1/4}}{\bar{\psi}_{10}}\bar{\phi}_1 \quad ; \qquad z=(\frac{k^2}{\Lambda})^{1/4}. \tag{84}$$

In terms of these variables, Eq.83 becomes

$$\frac{d^2\chi}{dz^2}-z^2\chi=z \tag{85}$$

and its solution can be obtained in terms of an integral representation [43]

$$\chi(z)=-z\int_0^{1/2}(1-4t^2)^{-1/4}e^{-z^2t}dt. \tag{86}$$

From this expression, it follows that the asymptotic behavior of $\chi(z)$ for large values of the argument is

$$\chi(z)\approx-\frac{1}{z}-\frac{2}{z^5}+... \quad ; \qquad z\gg 1. \tag{87}$$

4.2.3. *Asymptotic matching*

To match asymptotically the inner and outer solution to lowest order, it is sufficient to equate the jump in the logarithmic derivative of the former as $\bar{x}\rightarrow\pm\infty$ with Δ' given by Eq.73. The re-scaling of the variables inside the resistive layer was conveniently chosen such that the derivatives be independent of the scaling, i.e. $d\psi_1/dx=d\bar{\psi}_{11}/d\bar{x}$. Then using Eq.82, it follows that the discontinuity in the derivative of ψ_1 across the layer is given by

$$(\frac{d\psi_1}{dx})_+ - (\frac{d\psi_1}{dx})_- = -\frac{\Lambda^2}{k^2}\int_{-\delta}^{+\delta}\frac{d^2\bar{\phi}_1/d\bar{x}^2}{\bar{x}}d\bar{x}, \qquad (88)$$

where δ is the (normalized) half-width of the resistive layer. Considering that $\psi_1(0) = \psi_{10}$ =const, the jump in the logarithmic derivative is given by $\Delta_{int}' = (d\psi_1/dx\mid_+ - d\psi_1/dx\mid_-)/\psi_{10}$. Equating Δ_{int}' with Δ' obtained from the external solution, Eq.73, and writing the variables $\bar{x}$ and $\bar{\phi}_1$, in terms of z and χ, respectively, as defined in Eq.84, the equation for the eigenvalue Λ becomes

$$\Lambda^{5/4}k^{-1/2}\int_{-\delta}^{+\delta}\frac{dz}{z}\frac{d^2\chi}{dz^2} = \Delta'. \qquad (89)$$

From the asymptotic behavior of $\chi(z)$, Eq.87, it follows that the integrand in Eq.89 decreases with $1/z^4$ for large z. Therefore the limits of the integral can be taken to infinity, i.e.,

$$I \equiv \int_{-\delta}^{+\delta}\frac{dz}{z}\frac{d^2\chi}{dz^2} \approx \int_{-\infty}^{+\infty}\frac{dz}{z}\frac{d^2\chi}{dz^2} = \int_{-\infty}^{+\infty}dz\int_0^{1/2}\frac{(6t-4z^2t^2)}{(1-4t^2)^{1/4}}e^{-z^2t}dt, \qquad (90)$$

where the integral representation for $\chi(z)$, Eq.86, has been used. Carrying out the integration on z first and making the change of variable $\tau = 4t^2$, the integral I can be written in terms of the beta function $B(3/4, 3/4)$, which can be expressed in terms of Gamma functions [44]; the final result is $I = 2\pi\Gamma(3/4)/\Gamma(1/4)$. Substituting this result into Eq.89 and using Eq.78 to write the growth rate in terms of the actual physical quantities, the expression for γ becomes

$$\gamma = \left[\frac{\Gamma(1/4)}{2\pi\Gamma(3/4)}\Delta'\right]^{4/5}\epsilon^{3/5}(k^2a^2)^{1/5}\tau_A^{-1} = \left[\frac{\Gamma(1/4)}{2\pi\Gamma(3/4)}\Delta'\right]^{4/5}\tau_r^{-3/5}\left(\frac{\tau_A}{ka}\right)^{-2/5}, \qquad (91)$$

where Δ' is calculated from the solution of the linearized equations in the ideal region using Eq.73.

From the expression for the growth rate, (Eq.91), it is easy to see that the tearing mode is unstable, i.e., $\gamma > 0$, if $\Delta' > 0$. Actually, it can be shown that the energy available to drive the tearing mode is proportional to Δ' [32]. The basic picture of a tearing mode is then that magnetic energy released from the outer region is dissipated in the inner region [27–29].

The great strength of the approximate analysis using the boundary-layer technique is that the resistive equations do not have to be solved to determine the stability of an actual confinement configuration. Only the equation in the ideal region, Eq.66, has to be solved with proper boundary conditions; Δ' is then calculated from Eq.73 and substituted into Eq.91 to determine the growth rate. An illustrative example is given by choosing the normalized current density as $j(x) = sech^2(x)$ (Harris model), which peaks at $x = 0$ and vanishes away monotonically as $x \to \infty$. In this case the normalized perpendicular field is given by $B_y = f(x) = \tanh(x)$ and Eq.66 becomes

$$\frac{d^2\psi_1}{dx^2} - [k^2 - 2sech^2(x)]\psi_1 = 0. \quad (92)$$

The solution of this equation is given by

$$\psi_1(x) = e^{\pm kx}[1 \pm \frac{1}{k}\tanh x], \quad (93)$$

where the upper sign is for $x < 0$ and the lower for $x > 0$. Substituting this solution into Eq.73, it follows that

$$\Delta' = 2\left[\frac{1}{k} - k\right]. \quad (94)$$

From this expression it is clear that the current sheath is unstable for long wavelength tearing modes, i.e., $\Delta' > 0$ for $k < 1$. Although this result is strictly valid only for the particular equilibrium model that was considered, $j = sech^2(x)$, it remains qualitative correct for other equilibria, that is, the most unstable tearing modes have long wavelength in comparison with the width of the current layer.

5. Tearing Mode in a Cylindrical Plasma Column

To study tearing modes in magnetic confinement configurations that are relevant for fusion research, such as tokamaks, it is necessary to consider at least a cylindrical equilibrium model. Actually tokamaks are toroidal devices and even a cylindrical model is only an approximation to the real magnetic configuration. However, the driving force of tearing modes, which is proportional to the gradient of the equilibrium current density, is much stronger than the restoring force that results from the toroidal curvature of the field lines [6] and they can be analysed to a good approximation in cylindrical geometry. The main effect of toroidicity is to introduce a small positive threshold on the value of Δ' for the modes to become unstable [22,45].

As in the slab model, the stability of tearing modes is analysed considering two regions, an ideal region wherein the effect of resistivity is neglected and a resistive layer around the singular points of the linearized ideal equation for the magnetic field perturbation. In the slab model, the singular point occurs at $x = 0$, where the perpendicular component of the equilibrium magnetic field vanishes. Since the mode wavevector is considered in the y-direction, this is also the point where $\vec{k} \cdot \vec{B} = 0$, that is, the point where the restoring force due to the bending of the field lines vanishes, as discussed in section 2. The same is true in cylindrical geometry; however in this case the equilibrium quantities depend on r and the ignorable coordinates are θ and z. The perturbed quantities are the expressed in the form

$$\psi_1(r,\theta,z;t) = \psi_1(r)\exp[\gamma t - i(\frac{m}{r}\theta + kz)], \quad (95)$$

where the wavevector is given by $\vec{k} = (m/r)\hat{e}_\theta + k\hat{e}_z$. The equilibrium magnetic field is given by $\vec{B} = B_\theta(r)\hat{e}_\theta + B_z(r)\hat{e}_z$ and therefore the positions $r = r_s$ of the mode rational surfaces, where $\vec{k} \cdot \vec{B} = 0$, are given by the zeros of the function

$$F(r) = \frac{B_\theta}{r}[m - nq(r)]. \tag{96}$$

To write $F(r)$ in this form, the component of the wavevector in the z-direction was written as $k = -n/R$, what corresponds to impose periodic boundary conditions that emulate standing waves in a torus of major radius R [3–6]. The quantity $q = rB_z/RB_\theta$ is called the safety factor and m and n are called the poloidal and toroidal mode numbers, respectively.

From the analysis in the plasma slab, it is expected that the most unstable modes have long wavelengths, which corresponds to low values of n in cylindrical geometry; typically $n = 1, 2$. The conditions for ideal MHD stability, on the other hand, impose that the value of q be larger than two at the plasma boundary, $r = a$, and close to one at the magnetic axis, $r = 0$ [6]. Thus,the condition $F(r) = 0$ occurs for $m/n = 1/1; 3/2; 2/1; 5/2; 3/1$, etc. The mode with poloidal mode number $m = 1$, which is called the resistive internal kink mode, is a special case because it can be unstable already in the ideal MHD limit [6]. The mode growth rate can be determined using again the boundary-layer technique but the matching condition is somewhat different from the one for tearing modes. The theory of this mode will not be discussed in these lectures; the interested reader can find the necessary material in the original work on this instability [21] or in the excellent review of White [32]. Considering therefore only tearing modes with $m \geq 2$, the positions of the rational surfaces occur somewhere away from the axis of the plasma column. Since the width of resistive layer is of the order of $\epsilon^{2/5} \ll 1$, as follows from the scaling given by Eqs.74 and 81, the radius of a mode rational surface is much larger than the width of the resistive layer. Thus geometrical effects are not dominant in the resistive layer and the linearized equations including resistivity can be solved using the slab model approximation. The relevant equations are again Eqs.82 and 83 and the solution is given by Eqs.84 and 86. The mode growth rate is also given by the same expression for the slab model, Eq.91. The jump in the logarithmic derivative of $\psi_1(r)$ in the ideal MHD region is, however, not calculated from the solution of Eq.66. This is because the ideal MHD region runs from $r = 0$ to $r = r_s$ and from $r = r_s$ to $r = a$. In this wide domain the effects of the cylindrical geometry are certainly relevant and have to be properly taken into account. The ideal MHD perturbed displacements in cylindrical geometry are solutions of the Newcomb equation

$$\frac{d}{dr}\frac{r^3F^2}{m^2 + k^2r^2}\frac{d\xi}{dr} - g(r)\xi = 0, \tag{97}$$

subject to the boundary conditions $\xi \sim r^{m-1}, r \to 0$, and $\xi(a) = 0$, for $m \geq 2$ [1,3–5]. In this equation ξ is the radial component of the perturbed fluid displacement, i.e., $\xi = v_{1r}/\gamma$, and

$$g(r) = \frac{m^2-1}{m^2+k^2r^2} rF^2 + \frac{k^2r^2}{m^2+k^2r^2}\left[2\mu_0\frac{dp}{dr} + rF^2 - 2rF\frac{(m/r)B_\theta - kB_z}{m^2+k^2r^2}\right]. \quad (98)$$

In order to use the results derived in the slab model, it is necessary to relate ξ to the perturbed magnetic flux function ψ_1 and obtain an equation for ψ_1, which will replace Eq.66 in cylindrical geometry. Naturally, because $\nabla \cdot \vec{B} = 0$, the magnetic field can still be written in the form of Eq.44, i.e.,

$$\vec{B} = \nabla\psi \times \hat{e}_z + B_z\hat{e}_z, \quad (99)$$

where now $\psi = \psi(r, \theta, z; t)$. The perpendicular component of the perturbed magnetic field is therefore given by

$$\vec{B}_{\perp 1} = \nabla\psi_1 \times \hat{e}_z = -\frac{d\psi_1}{dr}\hat{e}_\theta + i\frac{m}{r}\psi_1\hat{e}_r. \quad (100)$$

Linearizing Eq.5 in cylindrical geometry and considering $\eta = 0$, the equation for the perturbed magnetic field $\vec{B}_1$ in terms of the perturbed fluid displacement $\vec{\xi}$ can be readily obtained, viz.,

$$\vec{B} = iF\vec{\xi} - (r\frac{d}{dr}\frac{B_\theta}{r})\xi\hat{e}_\theta - (\frac{dB_z}{dr})\xi\hat{e}_z. \quad (101)$$

Comparing the radial components of Eqs.100 and 101, it follows immediately that

$$\psi_1 = \frac{1}{m} rF\xi. \quad (102)$$

Substituting this relation into Eq.97, the equation for ψ_1, becomes

$$\frac{d}{dr}\frac{r}{m^2+k^2r^2}\frac{d\psi_1}{dr} - \left[\frac{g}{(rF)^2} + \frac{1}{(rF)}\frac{d}{dr}\frac{r}{m^2+k^2r^2}\frac{d}{dr}(rF)\right]\psi_1 = 0. \quad (103)$$

This equation is the equivalent in cylindrical geometry to Eq.66 in slab geometry (note, however, that physical quantities are not yet normalized). It is clear that the points where $F(r_s) = 0$ are singular points of this equation. Although looking somewhat complicated, Eq.103 can be easily numerically integrated from $r = 0$ towards $r = r_s$ and from $r = a$ towards $r = r_s$. Close to the singular point, it can be shown that it assumes the form of Eq.67, where the variable x is replaced by $(r - r_s)/a$ and the constant K becomes somewhat more involved [20,46]. The asymptotic behavior close to the singularity is therefore still given by Eqs.69,79, and 71. Normalizing ψ_1, F, and g as described in sub-section 4.1 and defining $x = (r - r_s)/a$, the numerical solutions in the left ($r < r_s$) and right ($r > r_s$) intervals can then be approximated as

$$\psi_1(x < 0) = \psi_1(0)\psi_s(x) + A_1\psi_r(r)$$

and

$$\psi_1(x > 0) = \psi_1(0)\psi_s(x) + A_2\psi_1(x), \tag{104}$$

where $\psi_1(0), A_1$, and A_2 are constants. Once these constants are determined by fitting the numerical solutions, Δ' can be calculated using Eq.73

$$\Delta' = \psi_a'(0_+) - \psi_s'(0_-) + \frac{A_2}{\psi_1(0)}\psi_r'(0_+) - \frac{A_1}{\psi_1(0)}\psi_r'(0_-). \tag{105}$$

From Eqs.70 and 71 it follows that $\psi_s'(0_+) \to \psi_s'(0_-)$ and $\psi_r'(0_\pm) \to K$; therefore

$$\Delta' = \frac{(A_2 - A_1)}{\psi_1(0)} K. \tag{106}$$

The mode growth rate is then calculated substituting this value of Δ' into Eq.91.

In practical applications for tokamaks, Eq.103 is usually simplified taking into consideration that $(kr/m)^2 = (nr/mR)^2 \ll 1$. In this case, the function g becomes to lowest order $g \approx (m^2 - 1)rF^2/m^2$ and Eq.103 can be simplified to

$$\frac{1}{r}\frac{d}{dr}r\frac{d\psi_1}{dr} - \frac{m^2}{r^2}\psi_1 - \frac{m\mu_0(dj/dr)}{B_\theta(m - nq)}\psi_1 = 0, \tag{107}$$

where j is the component of the current density in the z-direction. From this equation one can see that close to the singular point, $m - nq(r_s) = 0$, the equation indeed becomes of the form of Eq.67, with the constant K given by

$$K = -\frac{\mu_0 ma(dj/dr)_{rs}}{n(B_\theta dq/dr)_{rs}}. \tag{108}$$

Thus the linear stability of tearing modes ($\Delta' > 0$ or $\Delta' < 0$) depends on the gradient of the current density and on the shear (dq/dr) of the equilibrium configuration at the mode rational surfaces.

The stability of tearing modes in cylindrical geometry can therefore be investigated by just solving (usually numerically) the ideal MHD eigenvalue equation, Eqs.103 or 107. Furth, Rutherford, and Selberg carried out a detailed inestigation of stability of tearing modes as a function of the shape of the current profile [20]. They found that only the $m = 2, 3$ tearing modes are usually unstable for peaked profiles and that larger m modes can become progressively unstable as the current profile is broadened. A very useful stability diagram for cylindrical plasma columns, combining ideal and resistive modes for different current profiles, was later produced by Wesson [33]. To investigate the stability of toroidal plasma columns taking full account of toroidal effects is much more involved because of the linear coupling between a mode with mode number m and its satellites, $m - 1$ and $m + 1$ [45]. However, most of the experimental results can be reasonably explained by the cylindrical model [23,28,40].

6. A Pedestrian Description of the Non-Linear Evolution of Tearing Modes

The main effect of tearing modes is to produce current filamentation in the form of magnetic islands. The width of these islands is proportional to the square root of the perturbed magnetic flux, as it will be shown below, and therefore it grows exponentially in the linear regime. Soon the island width becomes larger than the width of the resistive layer and nonlinear effects start to dominate over inertia [7,8]. The further growth of the islands proceeds still driven by the available magnetic energy, which is proportional to the current density, as in the linear regime. However, the growth is limited primarily by resistive diffusion and not by fluid inertia. In this section a brief and simplified description of the non-linear evolution of tearing modes is presented; the interested reader will find the detailed theory described in references 7,8,27, and 47.

6.1. *Magnetic Islands*

In the two-dimensional geometry considered in sub-section 4.1, the equation for a field line is given simply by $dx/B_x = dy/B_y$. Then, it follows immediately from Eq.44 that the $d\psi = 0$ along a field line, i.e., the field lines lie on $\psi = const$ surfaces. The total flux function, including the equilibrium and perturbed contributions, is given by (Eq.53)

$$\psi(x,y;t) = \psi_0(x) + \psi_1(x;t)\cos ky, \tag{109}$$

where, in the *linear phase*, $\psi_1(x;t) = \psi_1(x)e^{\gamma t}$. For a fixed time, the surfaces $\psi = const$ have the form shown in Fig.4. A critical surface, called the *separatrix*, separates the field lines that preserve the topology of the equilibrium from those that spiral on closed nested surfaces. This configuration is called a *magnetic island*. The width of a magnetic island can be calculated from the equation for the separatrix.

The stagnation points of the curves $\psi = const$ are given by the solutions of the equation $\partial\psi/\partial x = 0$ and $\partial\psi/\partial y = 0$. Since $\partial\psi_0/\partial x = 0$ at $x = 0$ and $\psi_1(x) = \psi_{10} = const$ to lowest order, the stagnation points are given by $x = 0$; $y = n\pi/k$; $n = 0, 1, 2, ...$ Calculating $D \equiv (\partial^2\psi/\partial x^2)(\partial^2\psi/\partial y^2) - (\partial^2\psi/\partial x\partial y)^2$, it can be seen that n even corresponds to elliptical stagnation points ($D > 0$) and n odd to hyperbolic stagnation points ($D < 0$). The former are thus the magnetic axes of the islands and the latter are the X-points of the separatrix. The value of ψ at the separatrix is then given by $\psi(0, \pi/k) = \psi_0(0) - \psi_1(0)$. Thus, the width of the island is determined from the points where the curve $\psi(x, y) = \psi_0(0) - \psi_1(0)$ crosses the $y = 0$ axis, i.e.,

$$\psi_0(x) + \psi_1(x) = \psi_0(0) - \psi_1(0). \tag{110}$$

Expanding $\psi_0(x)$ in a power series and recalling that $(d\psi_0/dx)_0 = 0$ and $\psi_1(x) \approx \psi_{10} = const$, it follows from Eq.110 that

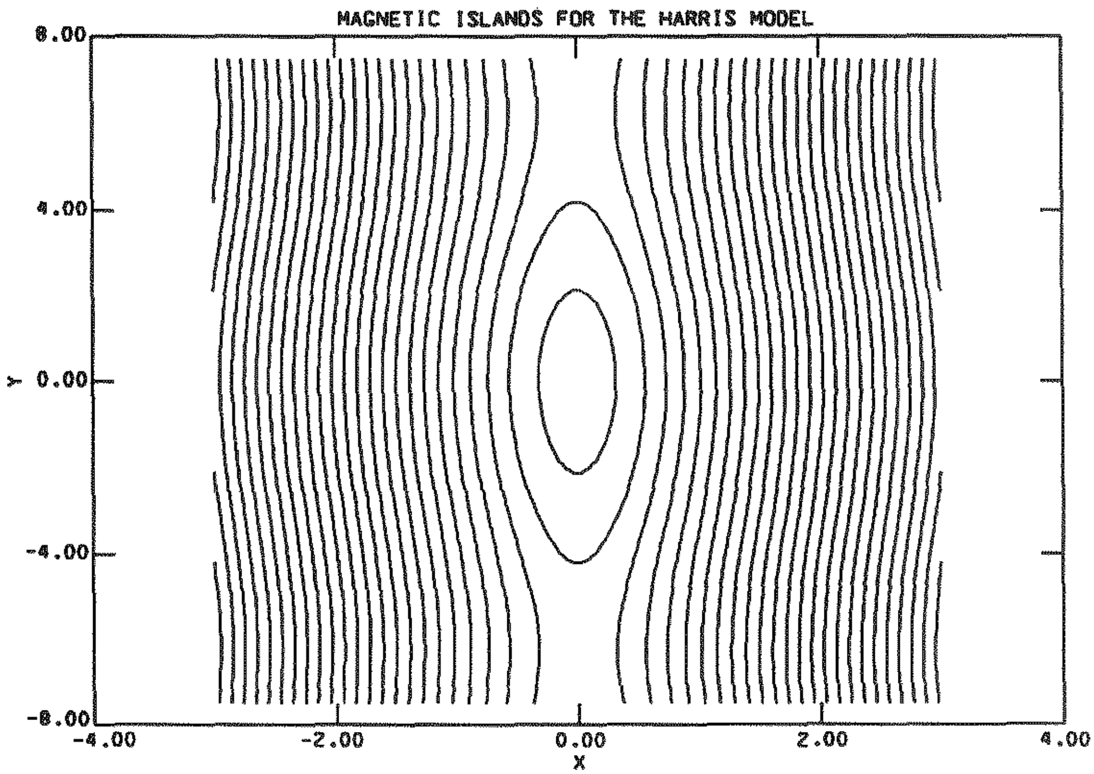

Figure 4: Magnetic island configuration. In this case $k = 0.5; \psi_{10} = 0.1$.

$$\psi_0(0) + \frac{1}{2}(\frac{d^2\psi_0}{dx^2})_0 x^2 + ... + \psi_{10} = \psi_0(0) - \psi_{10}; \tag{111}$$

thus, the island width, $w = x_+ - x_-$ is given by

$$w = 4\left[\frac{-\psi_{10}}{(d^2\psi_0/dx^2)_0}\right]^{1/2}. \tag{112}$$

Considering the time variation in the linear regime, $\psi_1(x;t)$ increases exponentially, i.e., $\psi_1(x) \approx \psi_{10}e^{\gamma t}$, and therefore the island width also has an exponential increase. In the nonlinear regime $\psi_1(x;t)$ does not vary exponentially with time, by it can still be considered approximately constant across the island [7,8]. In this case, the expression for the width of the magnetic island is given by Eq.112 with ψ_{10} replaced by $\psi_1(0;t)$.

6.2. *Quasilinear Theory*

The simplest theory for the nonlinear evolution of magnetic islands can be derived by assuming that the physical quantities are given by an average plus a fluctuating part, i.e.,

$$\psi =< \psi > +\tilde{\psi}, \tag{113}$$

where $< \psi >$ is an ensemble average and $\tilde{\psi}$ is not necessarily small. Restricting this derivation to the slab model and to modes propagating in the y-direction, the fluctuation can be Fourier analysed,

$$\tilde{\psi}(x,y;t) = \sum_{n\neq 0} \psi_n(x,t)e^{inky} + comp.conj., \tag{114}$$

and the ensemble average can be correspondingly defined as

$$< \psi >= \frac{k}{2\pi}\int_0^{2\pi/k} \psi(x,y;t)dy, \tag{115}$$

such that $< \tilde{\psi} >= 0$. The correlation between two fluctuating quantities, say $\tilde{\psi}$ and $\tilde{\phi}$, is then given by

$$< \tilde{\psi}\tilde{\phi} >= \sum_{n=0}[\psi_n(x,t)\phi_n^*(x,t) + \psi_n^*(x,t)\phi_n(x,t)]. \tag{116}$$

Substituting Eq.113 into Eq.55 and considering $< \vec{v} >= 0$ yields

$$\frac{\partial}{\partial t}(< \psi > +\tilde{\psi}) + (\tilde{\vec{v}}\cdot\nabla)(< \psi > +\tilde{\psi}) = \epsilon\nabla^2(< \psi > +\tilde{\psi}) - E. \tag{117}$$

The ensemble average of this equation gives

$$\frac{\partial < \psi >}{\partial t} + < \tilde{\vec{v}}\cdot\nabla\tilde{\psi} >= \epsilon\nabla^2 < \psi > -E. \tag{118}$$

Subtracting Eq.118 from Eq.117 it follows that

$$\frac{\partial\tilde{\psi}}{\partial t} + \tilde{\vec{v}}\cdot\nabla < \psi > -\epsilon\nabla^2\tilde{\psi} = [\tilde{\vec{v}}\cdot\nabla\tilde{\psi} - < \tilde{\vec{v}}\cdot\nabla\tilde{\psi} >]. \tag{119}$$

The term on the righ-hand-side of this equation is due to the coupling between different modes. If there are two fluctuations with mode numbers n_1 and n_2, the quadratic term will give origin to fluctuations with mode numbers $2n_1, 2n_2$, and $n_1 \pm n_2$. However, no average term ($n = 0$) will be induced because the ensemble average is subtracted from the quadratic term. The effect of the correlation between fluctuations on the evolution of the average magnetic flux is given by the term $< \tilde{\vec{v}}\cdot\nabla\tilde{\psi} >$ in Eq.118. The *quasilinear approximation* consists in neglecting the mode coupling term in Eq.119; it therefore becomes entirely equivalent to Eq.59 for the linear perturbation. However, the effect of the correlation between fluctuations on the evolution of the blackground magnetic flux function $< \psi >$ is retained.

The same procedure can be applied to the equation for the evolution of vorticity, Eq.56. In fact, the quasilinear theory for tearing modes can be self-consistently carried out in slab geometry. The main result is that a spectrum of unstable tearing modes give rise to an anomalous resistivity that is always positive and therefore increases the rate of flux diffusion [48]. The full theory will not be discussed in these lectures; instead, using Eqs.118 and 119, a crude derivation of the expression for the quasilinear evolution of magnetic islands will be presented. Neglecting the mode coupling term in Eq.119, writing $\tilde{v} = \nabla\tilde{\phi} \times \hat{e}_z$, and using the scaling relations Eqs.74 to 78, it follows that the first and third terms are of order $\epsilon^{-1/5}$. Therefore, to lowest order the second term has to vanish, i.e., the variation of $< \psi >$ across the island has to be much smoother than that of $\tilde{\psi}$ (this argument is qualitatively valid only if the island width is not much larger than the width of the resistive layer). To next order the first and third terms have to balance, i.e.,

$$\frac{\partial\tilde{\psi}}{\partial t} \approx \epsilon\frac{\partial^2\tilde{\psi}}{\partial x^2}, \tag{120}$$

where it has been assumed that the wavelength is much larger than the island width, ($k^2w^2 \ll 1$) such that $\nabla^2\tilde{\psi} \approx \partial^2\tilde{\psi}/\partial x^2$. Considering that only the derivatives of $\tilde{\psi}$ are discontinuous at the island, the second derivative can be approximated as $\partial^2\tilde{\psi}/\partial x^2 \approx [d\tilde{\psi}/dx \mid_+ - d\tilde{\psi}/dx \mid_-]/w \approx \Delta'(w)\tilde{\psi}/w$, where w is the island width and $\Delta'(w)$ is defined as in Eq.73 but with the derivatives calculated at the edges of the island, i.e., $(d\tilde{\psi}/dx)_\pm = (d\tilde{\psi}/dx)_{\pm w/2}$. Then Eq.120 becomes

$$\frac{\partial\tilde{\psi}}{\partial t} \approx \epsilon\frac{\Delta'\tilde{\psi}}{w}. \tag{121}$$

Finally, using the expressions for the island width, Eq.112, with ψ_{10} replaced by $\tilde{\psi}$, the above equation can be written as

$$\frac{dw}{dt} \approx \frac{1}{2}\Delta'(w)\epsilon. \tag{122}$$

This equation shows that in the quasilinear regime the island grows on the resistive time scale [the non-normalized form of the equation is $dw/dt = 0.5(a\Delta')(a/\tau_r)$]. This growth is however not linear because Δ' is usually a monotonically decreasing function of w. The island saturates when $\Delta'(w) = 0$, i.e., when the outside magnetic energy that drives the linear tearing mode vanishes. For the Harris slab model, i.e., $j(x) = sech^2(x)$, $\Delta'(w)$ can be readily calculated from the solution of the ideal MHD equation (Eq.93); one obtains

$$\Delta'(w) = e^{-(kw/2)}[\Delta'(0) - 2\tanh\frac{w}{2}(1 + \tanh\frac{w}{2})], \tag{123}$$

where $\Delta'(0) = 2(1/k - k)$. This function is plotted in Fig.5 for $k = 0.5$ (recall that k is normalized to a^{-1}). Since $\tanh x \to 1$ as $x \to \infty$, it follows that for this model the islands saturate ($\Delta' = 0$) only if $\Delta'(0) < 4$. For $k = 0.5$, it can be seem from the Fig.5 that the normalized width of the saturated island is quite large, namely, $w_s = 2.33$.

The expression for Δ' can be roughly approximated by a linear variation of the type $\Delta'(w) = \Delta'(0)(1 - w/w_s)$, where w_s is the saturation width. Substituting this linear expression into Eq.122, the time evolution of the island width becomes

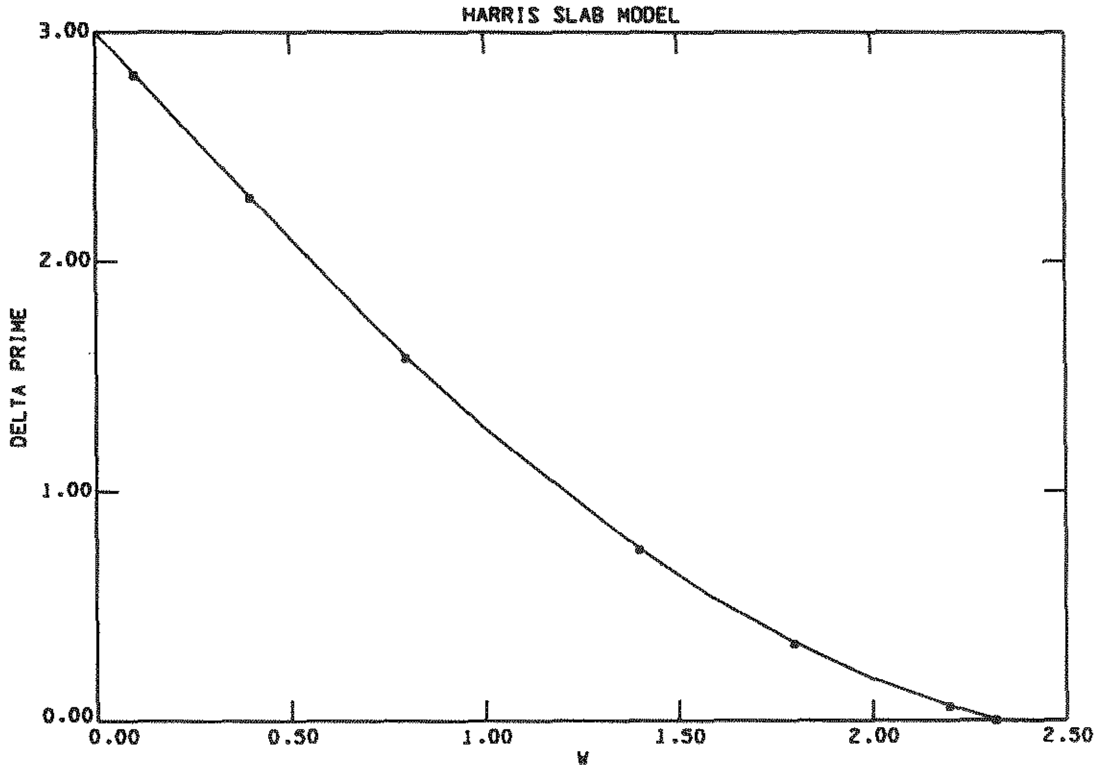

Figure 5: $\Delta'(w)$ for the Harris slab Model with $k = 0.5$.

$$\frac{w}{w_s} \approx 1 - e^{-\frac{1}{2}(\epsilon\Delta'(0)/w_s)t}. \tag{124}$$

To calculate $\Delta'(w)$ in cylindrical geometry, Eq.107 has to be integrated for ψ_1, stopping the integration at increasing distances from the singular point. Actually, in cylindrical geometry the equation for the island evolution is given by

$$\frac{dw}{dt} = 1.66\epsilon[\Delta'(w) - \alpha w], \tag{125}$$

where the numerical factor comes from a proper consideration of inertial effects and the curvature of the field lines and α is a constant that depends on the resistivity profile [8,32]. Saturation then occurs for $\Delta'(w) = \alpha w$; however, for usual experimental profiles, α can be neglected and the saturation width is approximately given by the solution of $\Delta'(w) = 0$.

6.3. *Experimental Observations*

Experimentally the islands grow quickly and only saturated islands can be normally observed. The current density is approximately uniform inside the islands; therefore they show up as helical current filaments at the mode rational surfaces producing a perturbed magnetic field that can be detected outside the plasma column. Usually the islands rotate; the rotation is either due to a global rotation of the plasma column, caused by ambipolar fields [49], or to kinetic effects that produce overstable modes with a real frequency close to the electron diamagnetic frequency [50]. The two processes cannot be easily distinguished experimentally because they both predict rotation of the same order of magnitude and in the same direction [4]. Nevertheless, the rotating perturbed field produced by the islands is observed in external pick-up coils as magnetic fluctuations, called Mirnov oscillations [51]. The amplitude of the Mirnov oscillations can be reasonably predicted by the theory of saturated islands in cylindrical geometry, as shown by Carreras, Waddell, and Hicks [52]. The rotation of the islands can be completly stopped by supperposing on the plasma external helical fields that resonantly interact with current inside the islands [53] or by the perturbed current induced in the vacuum vessel and conducting structures surrounding the plasma column [54]. In both cases the experimental observations can also be explained by the quasilinear theory of island growth. Finally, if two islands in neighboring mode rational surfaces grow up to the point of touching each other, a strong stochastic field configuration may result, leading to the disruption of the plasma column [23]. This picture is consistent with the experimental observation of high-current disruptions in JET [55,56].

7. Conclusion

An introduction to the theory of resistive modes in magnetized plasmas has been presented in these lectures at the graduate student level. Inevitably, many relevant topics have been omitted; some of them must be included in a standard course on resistive modes and others are part of the current research in this area. A few examples of the former are the theory of the resistive internal kink mode, kinetic effects, toroidal effects, and forced magnetic reconnection. Examples of the latter are the theory of locked modes, transport induced by self-sustained magnetic islands, and the dynamics of minor and major disruptions in tokamaks. Obviously, it would be impossible to cover all this material in four lectures. Instead, the choice has been to discuss only the basic physics and mathematical techniques that are essential to start in this subject. The interested student can proceed to more advanced topics beginning with the references that have been given.

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