Kruskal - Shafranov criterion

by Vinícius Duarte

Reference: B. B. Kadomtsev, in "Reviews of Plasma Physics", vol. 2, pp. 153–158 and 174–179 (Consultants Bureau, 1966).

Introduction to the hydromagnetic stability of a plasma:

Let us start with the linearly perturbed MHD eqs. in the absence of flows:

$$\frac{\partial \rho'}{\partial t} + \nabla\cdot(\rho_0 \underline{v}') = 0 \qquad \text{(Continuity eq.)} \qquad (1)$$

$$\rho_0 \frac{\partial \underline{v}'}{\partial t} + \nabla p' = \left(\frac{\nabla\times \underline{B}_0}{\mu_0}\right)\times \underline{B}' + \left(\frac{\nabla\times \underline{B}'}{\mu_0}\right)\times \underline{B}_0 \qquad \text{(Momentum conservation eq. + Ampère law)} \qquad (2)$$

$$\rho_0\left(\frac{\partial p'}{\partial t} + \underline{v}'\cdot\nabla p_0\right) = \gamma p_0\left(\frac{\partial \rho'}{\partial t} + \underline{v}'\cdot\nabla \rho_0\right) \qquad \text{(Adiabatic eq. of state)} \qquad (3)$$

$$\frac{\partial \underline{B}'}{\partial t} = \nabla\times(\underline{v}'\times \underline{B}_0) \qquad \text{(Faraday law + ideal Ohm law)} \qquad (4)$$

Instead of solving the eqs. for $\underline{v}'$, it's more convenient to introduce a fluid displacement $\underline{\xi}$ with respect to the equilibrium position, in such a way that $\underline{v}' = \frac{\partial \underline{\xi}}{\partial t}$ (Eulerian picture). If the equilibrium quantities have no time dependence, the eqs. can be integrated in time. Eq. (1) becomes

$$\rho' = -\nabla\cdot(\rho_0 \underline{\xi}) \qquad (5)$$

that, substituted into (3), gives

$$p' = -\underline{\xi}\cdot\nabla p_0 - \gamma p_0 \nabla\cdot\underline{\xi} \qquad (6)$$

Similarly, (4) becomes

$$\underline{B}' = \nabla \times (\underline{\xi} \times \underline{B}_0) \tag{7}$$

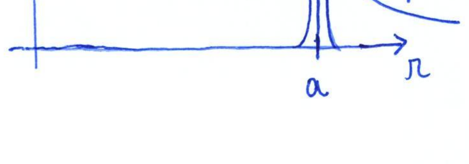
Substituting (5), (6) and (7) into (2) leads to a 2nd order differential eq. for $\underline{\xi}$ in terms only of unperturbed quantities:

$$\rho_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = \nabla\left[\underline{\xi}\cdot\nabla p_0 + \gamma p_0 \nabla\cdot\underline{\xi}\right] + \frac{1}{\mu_0}\left\{(\nabla\times\underline{B}_0)\times\left[\nabla\times(\underline{\xi}\times\underline{B}_0)\right] + \left[\nabla\times\nabla\times(\underline{\xi}\times\underline{B}_0)\right]\times\underline{B}_0\right\} \tag{8}$$

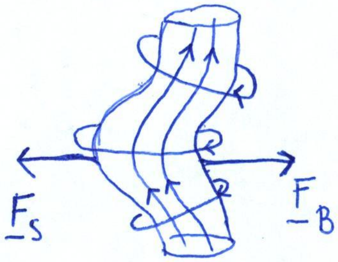
Under suitable boundary conditions, the RHS of eq. (8) can be shown to be self-adjoint, which allows an energy principle [see Bernstein et al, Proc. Royal Soc, 244A (1958) 17] to be written. By assuming $\underline{\xi}(\underline{r},t) = \underline{\xi}(\underline{r})e^{-i\omega t}$, we then have an eigenvalue problem for ω^2. If all possible ω^2 are positive, the equilibrium is stable. If at least one eigenvalue ω^2 is negative, the perturbation grows exponentially in time and the equilibrium is unstable.

Surface-layer pinch in a longitudinal magnetic field (skin pinch)

Let us study the stability of a cylindrical, linear pinch in the presence of a constant background field in the longitudinal direction. The currents are assumed to flow only in a thin surface sheet so that the pressure p_0 and the longitudinal field B_z are independent of r. This example will illustrate the phenomenon of stabilization via magnetic shear.



Let us consider a plasma column that was bent due to the action of a perturbation. We want to know if it will evolve to an instability or if it will be stabilized.



Note that, in the inner side of the kink, the azimuthal field pressure is larger than in the outer side. This gives rise to a force $\underline{F}_S$ that points outwards. On the other hand, the stretching of axial field lines due to perturbation generates a restoring force $\underline{F}_B$ with opposite direction. If $|\underline{F}_B| \geq |\underline{F}_S|$, the column will be stabilized with respect to the kink.

Due to cylindrical symmetry, the perturbation can be taken as $\underline{\xi}(r,\theta,z,t) = \underline{\xi}(r)\exp[i(m\theta + kz - \omega t)]$ and ξ_z can be chosen in such a way as to make $\nabla\cdot\underline{\xi} = 0$ (incompressible fluid). So, eq. (7) can be written as

$$\underline{B}' = \nabla\times(\underline{\xi}\times\underline{B}_0) = \underline{\xi}\underbrace{(\nabla\cdot\underline{B}_0)}_{0} - \underline{B}_0\underbrace{(\nabla\cdot\underline{\xi})}_{0} + (\underline{B}_0\cdot\nabla)\underline{\xi} - (\underline{\xi}\cdot\nabla)\underline{B}_0$$

Inside the pinch; $\underline{\xi} \neq 0$, $\underline{B}_{0i} = B_{0iz}\hat{z}$ with $B_{0iz} = \text{const}$. Hence

$$(\underline{B}_{0i}\cdot\nabla)\underline{\xi} = B_{0iz}\frac{\partial\xi_r}{\partial z}\hat{r} + B_{0iz}\frac{\partial\xi_\theta}{\partial z}\hat{\theta} + B_{0iz}\frac{\partial\xi_z}{\partial z}\hat{z} = ikB_{0iz}\underline{\xi} \text{ and } (\underline{\xi}\cdot\nabla)\underline{B}_{0i} = 0$$

So, $\underline{B}_i' = ikB_{0i}\underline{\xi}$. We need to work out the terms that appear in eq. (8):

$$\nabla\times\nabla\times(\underline{\xi}\times\underline{B}_{0i}) = \nabla\times\underline{B}_i' = ikB_{0iz}\nabla\times\underline{\xi}$$

$$[\nabla\times\nabla\times(\underline{\xi}\times\underline{B}_{0i})]\times\underline{B}_{0i} = (\nabla\times\underline{B}_i')\times\underline{B}_{0i} = \nabla(\underline{B}_{0i}\cdot\underline{B}_i') - \underline{B}_i'\times\underbrace{(\nabla\times\underline{B}_{0i})}_{0} - (\underline{B}_{0i}\cdot\nabla)\underline{B}_i' - \underbrace{(\underline{B}_i'\cdot\nabla)\underline{B}_{0i}}_{0} =$$

$$= \nabla(\underline{B}_{0i}\cdot\underline{B}_i') - B_{0iz}\frac{\partial}{\partial z}\underline{B}_i' = \nabla(\underline{B}_{0i}\cdot\underline{B}_i') - k^2B_{0iz}^2\underline{\xi}, \text{ since } \underline{B}_i' = ikB_{0iz}\underline{\xi}$$

Substituting this in (8) and using $\nabla\times\underline{B}_{0i} = 0$, we have

$$\left(-\omega^2\rho_0 + \frac{k^2B_{0i}^2}{\mu_0}\right)\underline{\xi} = \nabla\tilde{p}, \quad \text{where } \tilde{p} \equiv p' + \frac{\underline{B}_i'\cdot\underline{B}_{0i}}{\mu_0} \qquad (9)$$

Taking the divergence of (9) and using $\nabla \cdot \underline{\xi} = 0$,

$$\nabla^2 \tilde{p} = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \tilde{p}}{\partial r}\right) - \left(\frac{m^2}{r^2} + K^2\right)\tilde{p} = 0$$

$\Rightarrow \tilde{p}(r) = c_1 I_m(Kr) + c_2 K_m(Kr)$, where I_m and K_m are modified Bessel functions of order m. K_m diverges at the origin and therefore is not physically acceptable. Note that the exponential dependence on θ, z and t has been omitted in the expression for $\tilde{p}(r)$. The solution can be normalized with its value at the plasma edge ($r=a$):

$$\tilde{p}(r) = \tilde{p}(a)\frac{I_m(Kr)}{I_m(Ka)} \Rightarrow (\nabla \tilde{p})_r = \frac{\tilde{p}(a) K I_m'(Kr)}{I_m(Ka)} \qquad (10)$$

that can be substituted into (9) at $r=a$:

$$\xi_r(r=a) = \frac{\tilde{p}(a) K}{\omega^2 \rho_0 - \frac{K^2 B_i(a)}{\mu_0}} \frac{I_m'(Ka)}{I_m(Ka)} \qquad (11)$$

Outside the pinch we have $\nabla \times \underline{B}_e' = 0$ for the perturbed field. So there exists a scalar magnetic potential that satisfies $\nabla^2 \psi = 0$ and has the solution (for $r > a$).

$$\psi = C \frac{K_m(Kr)}{K_m(Ka)} e^{i(m\theta + Kz - \omega t)} \quad ; C \text{ const.} \qquad (12)$$

since $I_m(Kr)$ diverge for $r \to \infty$.

Now we need to impose boundary conditions at the plasma edge ($r=a$). From the pressure equilibrium:

$$p_0 + p' + \frac{(\underline{B}_{0i} + \underline{B}_i')^2}{2\mu_0} = \frac{(\underline{B}_{0e} + \underline{B}_e')^2}{2\mu_0} \qquad (13)$$

Such condition needs to be evaluated at the perturbed plasma-vacuum interface, where $\underline{r} = \underline{r}_0 + \xi_n \hat{n}$ and $\xi_n = \underline{\xi} \cdot \hat{n}$ is the normal displacement.

We will keep only up to the linear terms in the perturbation, e.g.,

$$p_0(\underline{r}_0+\underline{\xi}) \simeq p_0(\underline{r}_0)+\underline{\xi}\cdot\nabla p_0\big|_{r_0} = p_0(\underline{r}_0)+\xi_n\frac{\partial}{\partial n}p_0(\underline{r}_0) \quad \text{and} \quad p'(\underline{r}_0+\underline{\xi}) \simeq p'(\underline{r}_0)$$

So we obtain (all following quantities are evaluated at r_0):

$$p_0+\xi_n\frac{\partial p_0}{\partial n}+p'+\frac{1}{2\mu_0}\left[B_{0i}^2+\xi_n\frac{\partial B_{0i}^2}{\partial n}+2\,\underline{B}_{0i}\cdot\underline{B}_i'\right] = \frac{1}{2\mu_0}\left[B_{0e}^2+\xi_n\frac{\partial}{\partial n}B_{0e}^2+2\,\underline{B}_{0e}\cdot\underline{B}_e'\right]$$

Using eq. (6) and $p+B_{0i}^2/2\mu_0 = B_{0e}^2/2\mu_0$, we obtain

$$-\gamma\, p_0\,\nabla\cdot\underline{\xi}+\frac{\underline{B}_{0i}\cdot\underline{B}_i'}{\mu_0} = \frac{\underline{B}_{0e}\cdot\underline{B}_e'}{\mu_0}+\frac{\xi_n}{2\mu_0}\frac{\partial}{\partial n}\left(|\underline{B}_{0e}|^2-|\underline{B}_{0i}|^2\right) \tag{14}$$

Since $B_{0e\varphi}\sim 1/r$, it follows that

$$\frac{\partial}{\partial r}|\underline{B}_{0e}|^2 = \frac{\partial}{\partial r}\left(B_{0ez}^2+B_{0e\varphi}^2\right)\Big|_{r=a} = -2\,\frac{B_{0e\varphi}^2}{a} \tag{15}$$

When substituted into (14), it follows that

$$\tilde{p}(a) = \frac{\underline{B}_{0e}(a)\cdot\underline{B}_e'(a)}{\mu_0}-\frac{\xi_r(a)}{a\mu_0}B_{0e\varphi}^2(a) \tag{16}$$

The external perturbed field can be calculated from (12):

$$\underline{B}_e' = \left(k\,K_m'(kr)\,\hat{r}+\frac{im}{r}K_m(kr)\,\hat{\theta}+ik\,K_m(kr)\,\hat{z}\right)\frac{C}{K_m(ka)}\,e^{i(m\theta+kz-\omega t)} \tag{17}$$

(17) into (16) leads to

$$\tilde{p}(a) = \frac{i}{\mu_0}\left(\frac{m}{a}B_{0e\varphi}(a)+kB_{0ez}(a)\right)C-\frac{\xi_r(a)}{a\mu_0}B_{0e\varphi}^2(a) \tag{18}$$

A way of finding C is to project $\underline{B}_e' = \nabla\times(\underline{\xi}\times\underline{B}_{0e})$ onto $\hat{r}$ and use eq. (17):

$$\hat{r}\cdot\underline{B}_e' = \hat{r}\cdot\left[\nabla\times(\underline{\xi}\times\underline{B}_{0e})\right] \Rightarrow Ck\frac{K_m'(ka)}{K_m(ka)} = \frac{1}{a}\frac{\partial}{\partial\varphi}(\xi_r B_{0e\varphi})+\frac{\partial}{\partial z}(\xi_r B_{0ez}) = i\,\xi_r(a)\left(\frac{m}{a}B_{0e\varphi}(a)+kB_{0ez}(a)\right) \tag{20}$$

A dispersion relation can be found by using eqs. (11), (18) and (20) in order to eliminate $\xi_r(a)$, $\tilde{p}(a)$ and C in favor of ω^2:

$$\boxed{\mu_0 \rho_0 \omega^2 = K^2 B_{0iz}^2(a) - \left(\frac{m}{a} B_{0e\varphi}(a) + K B_{0ez}(a)\right)^2 \frac{K_m(Ka)\, I_m'(Ka)}{K_m'(Ka)\, I_m(Ka)} - \frac{K B_{0e\varphi}^2(a)}{a} \frac{I_m'(Ka)}{I_m(Ka)}} \quad (21)$$

Analysis of the terms of the dispersion relation — eq. (21):

(i) $K^2 B_{0zi}^2(a)$: restoring term due to the stretching of field lines (inside the pinch). It is always positive.

(ii) $-\left(\frac{m}{a} B_{0e\varphi}(a) + K B_{0ez}(a)\right)^2 \frac{K_m(Ka)}{K_m'(Ka)} \frac{I_m'(Ka)}{I_m(Ka)}$; restoring term due to stretching of field lines outside the pinch. It is always positive since $K_m(Ka)/K_m'(Ka) < 0$ and $I_m'(Ka)/I_m(Ka) > 0$. Note that when $\underline{K} = m/r\, \hat{\theta} + K\hat{z}$ is parallel to $\underline{B}_{oe}$, this term gives zero.

(iii) $-\frac{K B_{0e\varphi}^2(a)}{a} I_m'(Ka)/I_m(Ka)$; this term is always negative and is the one that can lead to instability. It's meaning is that magnetic pressure decreases as the distance from the pinch increases (remember eq. (18) and the $1/r$ dependence).

Pinch with $B_{ze} \gg B_{\varphi e}$

Assuming long wavelengths, $K_m'(Ka)/K_m(Ka) \simeq -m/Ka$ and $I_m'(Ka)/I_m(Ka) \simeq m/Ka$. In this case, eq. (21) takes the form

$$\mu_0 \rho_0 \omega^2 = K^2 B_{0iz}^2(a) + \left[K B_{0ez}(a) + \frac{m}{a} B_{0e\varphi}(a)\right]^2 - \frac{m}{a^2} B_{0e\varphi}^2(a) \quad (22)$$

Let us find the value of K that minimizes ω^2:

$$\frac{d}{dK}\left[K^2 B_{0iz}^2(a) + \left(K B_{0ez}(a) + m/a\, B_{0e\varphi}(a)\right)^2 - \frac{m}{a^2} B_{0e\varphi}^2(a)\right] = 0$$

that implies $K = -\frac{m}{a} \frac{B_{0ez}(a) B_{0e\varphi}(a)}{B_{0iz}^2(a) + B_{0ez}^2(a)}$. It is consistent with the assumption $Ka \ll 1$ if m is not large.

If $B_{0i} \simeq B_{0e}$, it is straighforward to note that only $m=1$ will lead to instability, since

$$\omega^2_{min} = \frac{B^2_{0e\varphi}(a)}{\mu_0 \rho_0 a^2}\left[\frac{m^2 B^2_{0iz}(a)}{B^2_{0ez}(a)+B^2_{0iz}(a)} - m\right] \qquad (23)$$

From eq. (22), in order to have $\omega^2 > 0$ it is necessary to have $|K| < \frac{B_{0e\varphi}(a)}{a B_{0ez}(a)}$.

In a periodic cylinder with $L = 2\pi R_0$, we have $|K| = |n|/R_0$, $n \in \mathbb{Z}^*$.

Therefore $\frac{B_{0e\varphi}(a)}{a B_{0ez}(a)} < |K| = \frac{|n|}{R_0}$

For the general case ($\forall n$):

$$\boxed{q(a) \equiv \frac{a B_{0ez}(a)}{R_0 B_{0e\varphi}(a)} > 1}$$

<u>Kruskal-Shafranov criterion</u>, derived independently by Kruskal & Schwarzchild [Proc. Royal Soc. A 223, 348 (1954)] and Shafranov [At. Energy. 5, 38 (1956)].