NONLINEAR WAVE EQUATIONS

Martin Kruskal

Department of Astrophysical Sciences, Princeton University, N.J. 08540

1. Prefatory Remarks

I wish to take this opportunity to thank Jürgen Moser and Battelle Seattle Research Center for the opportunity provided me to deliver a series of lectures during the summer 1974 on the subject of nonlinear dispersive wave equations, as well as to attend the other lectures and participate in the discussions that took place. However, writing up these notes presented the difficulty that much or most of the material I covered was not being presented for the first time, but has been written up before by me and my colleagues once or even many times before in various forms. Sometimes one feels it worthwhile to recast, modify, or enlarge upon what has previously been presented, but other times one has nothing further to write on a given subject, even though oral exposition to a virgin audience is still worthwhile. Thus it turned out that these notes are very very far from a balanced representation of my lectures; a much better idea of the material covered can be obtained from the references to the work of our group (mainly Clifford Gardner, John Greene, Robert Miura, Norman Zabusky, and myself) given in the bibliography, except that I omitted certain topics because they were being covered by various other lecturers (e.g. the Fermi-Pasta-Ulam problem, lectured on by Zabusky, and the method of exact solution of the KdV and other equations by reduction to a sequence of linear problems, lectured on by Alan Newell). Thus these notes cover a somewhat varied set of topics which I felt still deserved putting in a new light or at least giving more emphasis than they have received in our publications, together with enough background and explanatory

material to make them self-contained and give them, as I hope, a substantial degree of coherence. I refer the reader specifically to [Kruskal 1974] for some material I lectured on but do not include here.

I thank Robert Miura for making available to me his notes on my lectures.

2. Introductory Survey

My topic is nonlinear wave equations, and by a wave equation I mean first of all a partial differential equation (p.d.e.), like the classical linear wave equation

$$u_{tt} - u_{xx} = 0 \ . \tag{1}$$

(Subscripts denote partial differentiation.) But our attention will be restricted only to a special class of wave equations called conservative, or nondissipative, or reversible. Without attempting to delimit them precisely, we can say that they are p.d.e.'s that conserve some quantities which may be interpreted as energy, or entropy, rather than dissipating them, and that in fact are invariant with respect to the reversal of time, so that the initial value problems for them are no better (nor worse) posed than the "final value" problems. These properties presuppose that one of the independent variables (t) be singled out to represent time.

Thus we will not be considering equations such as the classical heat equation

$$u_t - u_{xx} = 0 \ , \tag{2}$$

which is nonconservative, dissipative, and irreversible (the initial value problem is well posed, if reasonable boundary conditions are prescribed, but not the final value problem).

Among the conservative nonlinear p.d.e.'s there are a few very special ones that in the last few years have been discovered to possess some quite remarkable properties. They have solitary wave solutions which, when brought to interact with one another (in a fully nonlinear way!), emerge completely unchanged in size and shape. They possess not just several conservation laws, like any run of the mill conservative p.d.e., but infinitely many conservation laws (of a local nature). They are each associated with a linear ordinary (time-independent) eigenvalue equation whose spectrum remains invariant under the evolution in t of coefficient functions, which involve the dependent variable of the original p.d.e. They represent completely integrable systems, as described by Jürgen Moser elsewhere in these notes. They can be solved (exactly) by reduction to a sequence of linear problems.

What are these special equations? The earliest known is the Korteweg-deVries [1895] (KdV) equation

$$u_t + uu_x + u_{xxx} = 0 \ . \tag{3}$$

Next is the "modified" or higher KdV equation

$$v_t + v^2 v_x + v_{xxx} = 0 \ , \tag{4}$$

which is closely related to the "standard" KdV equation (3), as we shall see. Then there is the so-called "sine-Gordon" equation [Rubinstein 1970],

$$\phi_{tt} - \phi_{xx} + \sin \phi = 0 \ , \tag{5}$$

which is a nonlinear version of the well known Klein-Gordon equation. Also the "nonlinear Schrödinger", or Zakharov-Shabat [1972], equation

$$i \psi_t + \psi_{xx} + \psi^2 \psi^* = 0 \ , \tag{6}$$

where $i = \sqrt{-1}$, asterisk denotes complex conjugate, and ψ is a complex variable (unlike the dependent variables in all the other cases, which are real). Finally, we have the recently discovered equation of Harry Dym [unpublished],

$$r_t = (r^{-1/2})_{xxx} , \tag{7}$$

again closely related to (3). Actually, each of these equations is merely the first (nontrivial) one in a sequence of increasingly higher order equations with closely linked properties. In very recent work, generalizations to other equations and even systems of equations have been found, as described by Alan Newell elsewhere in these notes.

Except for (7), all the equations above are quasilinear, i.e. linear in their highest derivative terms (where u_t , e.g., counts as a "highest derivative" term even though a third derivative term u_{xxx} is present, as long as no derivative term which would surely dominate it, such as u_{tt} or u_{xt} , is present). They are called "dispersive" wave equations, because the pure sinusoidal wave solutions of the corresponding linear equations (obtained by treating the dependent variable as small and neglecting higher order terms) disperse rather than travel together at a common phase velocity. Thus the linear equation corresponding to the KdV equation is

$$u_t + u_{xxx} = 0 , \tag{8}$$

which has sinusoidal solutions

$$u = \mathrm{Re}\ [\mathrm{const.}\ \exp\ (i\omega t + ikx)] \tag{9}$$

for k, ω satisfying the dispersion equation

$$\omega - k^3 = 0 . \tag{10}$$

Since the phase velocity

$$\frac{\omega}{k} = k^2 \tag{11}$$

depends on k (as does also the group velocity $d\omega/dk = 3k^2$), if a general localized initial function $u(x,0)$ is analyzed into sinusoidal modes, the solution of the linear problem at a later time will contain the same modes but at different relative positions, with different phase relations, and the solution $u(x,t)$ will be found to have "dispersed", i.e. spread out over a much wider interval than at $t = 0$, with essentially only one or a few discrete wavelengths apparent in any small local interval, though different ones in different intervals.

All this is in contrast to the standard linear wave equation (1), which is nondispersive, since its dispersion equation is

$$\omega^2 - k^2 = 0 \tag{12}$$

or

$$\omega = \pm k \ , \tag{13}$$

with two phase velocities

$$\frac{\omega}{k} = \pm 1 \tag{14}$$

(both) independent of k. Thus the general solution of (1) is

$$u(x,t) = f(x-t) + g(x+t) \ , \tag{15}$$

a superposition of two steady progressive waves travelling to the right and left without change of shape at velocities $\pm$ 1.

Note that despite the resemblance, the linear version of (5), the Klein-Gordon equation

$$\phi_{tt} - \phi_{xx} + \phi = 0 \ , \tag{16}$$

is dispersive because of the nonderivative term, its dispersion relation being

(17) $$\omega^2 = k^2 + 1$$

so that ω depends nonlinearly on k.

For the most part these very special equations have physical significance, and arise naturally in the study of physical problems. Thus the KdV equation was derived already in the preceding century as an approximate description of small amplitude water waves in a shallow channel (compared to the wavelengths of the waves). It was thereafter largely neglected until quite recently, although an excellent case could have been made for the importance of studying it on purely mathematical grounds. The KdV equation is, after all, the simplest nonclassical p.d.e.

The simplest p.d.e.'s have two independent variables. First order equations are well understood and are solvable by the method of characteristics. Second order equations are classified into elliptic, parabolic, and hyperbolic and are the backbone of classical mathematical physics. The simplest third order equation has a third derivative with respect to only one of the variables (x, say), and the minimum required to keep it from reducing to an ordinary differential equation is a first derivative with respect to the other variable (t). This gives the linearized KdV equation (8). But linear equations are in a certain sense trivial, at least those with constant coefficients. We might introduce a variable coefficient, most plausibly and simply x itself (this is in fact what Tricomi does to obtain his famous model equation with change of type from elliptic to hyperbolic), but spatial invariance has its own virtues and the term uu_x, nonlinear in the simplest way (quadratic), forms a natural and much precedented accompaniment to u_t, the sum of the two constituting the familiar convective (or substantial) derivative of fluid dynamics, if u is interpreted as a fluid velocity. (It might seem that u^2 would be a simpler term

to add, but uu_x provides an extra symmetry, Galilean invariance, as will be used later.)

In this way, one should, or could, or might have been led to study the KdV equation on its mathematical merits, but in fact neither that consideration nor its physical relevance sufficed to turn attention to it for a long time, till it resurfaced in modern times in connection with the Fermi-Pasta-Ulam [1955] problem. So much for historical necessity!

Now in fact it turns out that the KdV equation arises in a large number of physical situations. Besides water waves, it has been shown to apply to magnetohydrodynamic waves, to ion-acoustic plasma waves, anharmonic lattice vibrations, and several other situations. How is it possible that so ubiquitous an equation managed for so long to evade attention?

Well, actually there is something strange about the KdV equation: it is first order with respect to t differentiation. All the familiar fundamental equations of physics are of even order, mostly second order (if they involve t at all; the potential equation doesn't) -- with perhaps two notable exceptions. The reason is that the fundamental processes of physics are time-reversible, so the equations describing those processes should be invariant under the replacement of t by -t. This invariance is enjoyed by even derivatives but not odd ones. (It is less apparent why the square of the first time derivative should never occur in a fundamental equation, but it seems not to. Quasilinearity would tend to militate against such a term, but why should fundamental equations be quasilinear? And even quasilinearity would be retained if the second time derivative were also present. Of course, many fundamental equations are outright linear.)

The two exceptions merely emphasize the point. The heat (or

diffusion) equation (2) describes an irreversible process, and in fact is not considered truly fundamental in physics, since all situations to which it is known to apply have been analyzed into finer constituents satisfying reversible equations, so that (2) is always viewed as ultimately an approximation. The Schrödinger equation

$$i\psi_t + \psi_{xx} - u\psi = 0 \tag{18}$$

looks first-order in t but is really second order, since it amounts to two coupled first order equations, ψ being complex; it is even time-reversible if one replaces i by -i as well as t by -t (as why shouldn't one, since there is no intrinsic way to specify one root of -1 in preference to the other?).

3. Derivation of KdV Equation

Like the heat equation, the KdV equation arises always as an approximation to a higher order equation (or system of equations). The approximation scheme employed is (some version of) the multi-scale method, which has only been formalized systematically in modern times and thus led to widespread applications of the KdV equation.

As a simple illustration of its derivation, consider the following prototypical weakly nonlinear, weakly dispersive wave equation

$$y_{tt} = y_{xx}(1 + \varepsilon y_x) + \alpha y_{xxxx} , \tag{19}$$

where ε and α, measuring the nonlinearity and the dispersiveness respectively, are two comparably small parameters. (The linear

equation, with $\varepsilon = 0$, has dispersion relation $\omega^2 = k^2 - \alpha k^4$, so that ω/k is approximately constant for k not too large. If the relation is taken seriously even for k large, we should have $\alpha \leq 0$ to ensure well-posedness in the sense of Hadamard; otherwise ω can be negative imaginary and arbitrarily large, small wave-length perturbations can grow arbitrarily fast, and smooth initial data can become singular in arbitrarily short times or even immediately.)

Parenthetically, let me mention that (19) is actually just the lowest significant-order equation governing the evolution of the Fermi-Pasta-Ulam weakly nonlinear discrete mass lattice, governed by the equation

$$(20) \qquad \ddot{y}_n = h^{-2}(y_{n+1}-2y_n+y_{n-1})\ [1+\tfrac{1}{2}\varepsilon h^{-1}(y_{n+1}-y_{n-1})]\ ,$$

if the displacement function y_n and its time derivative $\dot{y}_n$ are assumed to discretely approximate smooth functions of $x = nh$, $y(x,t)$ and $y_t(x,t)$, and are Taylor expanded to fourth order in the (small) interparticle distance h. Here $\alpha = h^2/12$, since the terms $y_{xx} + \alpha y_{xxxx}$ together arise as the two leading nonvanishing terms of the expansion of $h^{-2}[y(x+h,t) - 2y(x,t) + y(x-h,t)]$, the elastic restoring force term for the lattice. [To be sure, this α is positive, but the Taylor expansion carried to the next nonvanishing term would give the linear equation $y_{tt} = y_{xx} + (h^2/12)y_{xxxx} + (h^4/360)y_{xxxxxx}$, with dispersion relation $\omega^2 = k^2 - \alpha k^4 + \frac{2}{5}\alpha^2 k^6$, so that ω^2/k^2 is positive definite and well-posedness is restored.]

To lowest order ($\varepsilon = \alpha = 0$), (19) becomes the linear wave equation whose solution is a superposition of right-progressing and left-progressing steady (shape-unchanging) "waves" (arbitrary functions), like (15). Let us seek a solution of the full equation (19) which is approximately just a right-progressing wave, not exactly steady of course, but at least only slowly varying. If we

change variables from x,t to ξ,t, where $\xi = x-t$, then y (informally using the same symbol for what is mathematically a different function of the new arguments) should now vary only slowly with t.

The change of variables is effected by the replacements $\partial/\partial x \to \partial/\partial\xi$, $\partial/\partial t \to \partial/\partial t - \partial/\partial\xi$, so $y_{tt} \to y_{tt} - 2y_{\xi t} + y_{\xi\xi}$ and (19) becomes (cancelling $y_{\xi\xi}$)

$$y_{tt} - 2y_{\xi t} = \varepsilon y_\xi y_{\xi\xi} + \alpha y_{\xi\xi\xi\xi} \ . \tag{21}$$

Since y now (i.e. expressed as a function of the new independent variables) varies slowly, by assumption, we set $\tau = \varepsilon t$, whence

$$\varepsilon^2 y_{\tau\tau} - 2\varepsilon y_{\xi\tau} = \varepsilon y_\xi y_{\xi\xi} + \alpha y_{\xi\xi\xi\xi} \ . \tag{22}$$

Dropping the relatively negligible ε^2 term, we arrive at essentially the KdV equation; indeed if $u = \frac{1}{2} y_\xi$, then

$$u_\tau + uu_\xi + \left(\frac{\alpha}{2\varepsilon}\right) u_{\xi\xi\xi} = 0 \ . \tag{23}$$

The remaining constant coefficient $\alpha/2\varepsilon$ could be transformed away by properly rescaling any two of the three variables u, τ, ξ. (If the $\alpha^2 y_{xxxxxx}$ term had been retained originally for the sake of well-posedness, it would have been dropped as higher order along with the term $\varepsilon^2 y_{\tau\tau}$, so the result would have been the same.)

4. Validity of the KdV Equation

At this point it is unfortunately necessary to divagate from straightforward exposition in order to indulge in a bit of polemical disputation. This is because in the last couple of years the validity of the KdV equation, as derived above and similarly elsewhere, has been challenged by T. Brooke Benjamin and his colleagues. In the following I shall be referring to and quoting from the article, "Model equations for long waves in nonlinear dispersive systems," by T. B. Benjamin, J. L. Bona, and J. J. Mahony [1972], which throws down the glove in unmistakable terms.

In order to exhibit the relevant arguments of Benjamin et al. and what I consider their essential flaws, it is preferable first to recast the derivation of the KdV equation given above into what may seem to be a simpler and more transparent form, though I believe its individual steps are not quite so well motivated. It has the advantage of never transforming the independent variables (which is, however, conducive to the misinterpretation of Benjamin et al., as will be seen).

Starting again with (19), we introduce right off

$$u \equiv y_x - y_t \,, \tag{24}$$

$$v \equiv y_x + y_t \,, \tag{25}$$

which are the so-called Riemann invariants of the lowest order (linear) approximation to (19), $y_{tt} = y_{xx}$. [This v, it should be said, has nothing to do with the modified KdV equation (4).] Noting that $y_x = (u + v)/2$, we calculate expressions for the time evolution of u and v, using (19) to eliminate y_{tt}. Thus

$$u_t = -u_x - \frac{u_x+v_x}{2} \varepsilon \frac{u+v}{2} - \alpha \frac{u_{xxx} + v_{xxx}}{2} \,, \tag{26}$$

$$v_t = v_x + \frac{u_x+v_x}{2} \varepsilon \frac{u+v}{2} + \alpha \frac{u_{xxx} + v_{xxx}}{2} \,. \tag{27}$$

If we assume that the initial disturbance in y was confined to a limited region, the right-going and left-going waves, represented by u and v respectively, soon separate and propagate each into regions where the other is absent, so that we can set v = 0 in (26) and obtain for u

$$u_t + u_x + \frac{\varepsilon}{4} uu_x + \frac{\alpha}{2} u_{xxx} = 0 \tag{28}$$

(and similarly of course for v). Thus the u and v equations have decoupled. [It should, however, not be thought that v = 0 is essential to obtain (28) -- the latter remains valid as a leading order approximation under not very severe assumptions on the form of v.]

Now (28) is nothing but the KdV equation, except written in the original independent variables x, t instead of being subjected to a Galilean transformation to eliminate the u_x term (and possibly some rescaling thereafter to adjust or normalize the coefficients). Furthermore, it is precisely in the form of equation (2.16) of Benjamin et al., aside from their use of capital letters, the presence of some inessential numerical coefficients, and the fact that their α is finite whereas mine is of order ε (and equal to $\alpha^2\varepsilon$ in their notation).

Benjamin et al. do not like some of the mathematical properties of the KdV equation, especially the behavior for large k of the dispersion function $\omega = \omega(k)$ of the linear version obtained by omitting the uu_x term. They therefore replace one x differentiation in the last term by a (negative) t differentiation, on the grounds that $u_x \approx - u_t$ by virtue of (28) itself, and so presumably $u_{xxx} \approx - u_{xxt}$. Accordingly they advocate the evolution equation

$$u_t + u_x + \frac{\varepsilon}{4} uu_x - \frac{\alpha}{2} u_{xxt} = 0 \ , \tag{29}$$

which, they say, though admittedly an approximation, with error $O(\varepsilon^2)$, is as valid as (28) but better behaved mathematically (for instance with respect to existence proofs). [Note the misprint just above their equation (2.18): the reference to (2.15) should obviously be to (2.16) instead.]

Before criticizing their espousal of (29), I must point out that the choice between (28) and (29) is no minor or side issue but the heart of their article. Indeed, just above their equation (2.18) they call the equivalence of what are written here as (28) and (29): "a point of central importance to our discussion". And they go on shortly to say, "It is worth further emphasis that, as an approximate model for long waves of small amplitude, [the two equations have] essentially the same formal justification."

Now for one thing we can dispute the supposed mathematical advantages of (29) over (28). Taking the equations in linearized form, introducing (9), and thinking of α as finite now (perhaps via rescaling), it comes to a comparison between the dispersion relations

$$\omega = -k + \frac{\alpha}{2} k^3 \quad , \tag{30}$$

$$\omega = -k/(1+\frac{\alpha}{2} k^2) \quad , \tag{31}$$

for (28) and (29) respectively. The former has phase and group velocities which can take either sign, are unbounded in magnitude, etc.; as discussed by Benjamin et al. under the heading, "Shortcomings of the KdV equation," they find these properties disagreeable on various mathematical and computational grounds. The latter, however, is beautifully behaved in these respects. But, and it is a big but, this is the case only if $\alpha > 0$ (excluding the trivial case $\alpha = 0$). For $\alpha < 0$, (31) is if anything even worse behaved than (30), since then ω becomes infinite already for finite k. And (19) must have $\alpha < 0$ to be well posed, as explained parenthetically

following it. It is true that Benjamin et al. purport to have $\alpha > 0$ in their case, which differs from (19), but if their approach were legitimate it should be expected to work in either case. Nevertheless, this is a relatively minor issue.

Another objection that might be raised is that (28), as derived above, is not an approximation but is exact, whereas (29) is at best only an approximation. Still, as mentioned before, in many cases (28) is also merely an approximation. Anyway, this too is not really an issue.

A more embarrassing criticism is to ask why Benjamin et al. stop with one such replacement. Why not replace uu_x by $-uu_t$? Or u_{xxx} in (28) by u_{xtt} or $-u_{ttt}$? This would be no less justifiable than what they do, but such a proposal begins to appear as a kind of reductio ad absurdum, since it strikes at the fundamental mathematical nature of the p.d.e. (e.g., more initial conditions at $t = 0$ would be needed to make a well posed problem).

What all this suggests is a crucial question which was inevitable right from the start. How can it be claimed that two distinct evolution equations are both satisfactory leading order approximations to the same given original "exact" equation? Since there is no suggestion that the equations could be equivalent (i.e. have the same solutions), and they obviously aren't, at most one of them can be correct. Properly performed complete systematic asymptotic reduction procedures lead to unique reduced equations--there can be no ambiguity. This is illustrated by the first (preferred) derivation given above, leading to the KdV equation (23). But the second derivation à la Benjamin et al. seems to offer the choice between (28) and (29), among many other possibilities. How can this be? It might be mentioned that the same question arises in the recent work of Neil Berger [1974] giving a rigorous deriva-

tion of the KdV equation. Again he can derive distinct forms of the equation with equal validity.

The obvious answer is that (28) and (29), etc., are not completely reduced leading order approximations. This is obvious because they explicitly contain ε and α. The point is obscured in the paper by Benjamin et al. because they emphasize what they call "the tidy form" of the equations, namely the equations they start right out with [their equations (1.1), (1.2)],

$$u_t + u_x + uu_x + u_{xxx} = 0 \ , \tag{32}$$

$$u_t + u_x + uu_x - u_{xxt} = 0 \ . \tag{33}$$

These are misleading because ε must be understood to occur implicitly in the definitions of the dependent and independent variables -- see their equation (2.17) and the sentence following it. What they have done is rescale the variables in (28) and (29) by certain powers of ε (and numbers) to obtain (32) and (33).

The essential significance of the KdV equation in the present context is that it is the unique leading order reduced equation approximation. In (28), we are to treat u, x, and t as finite, i.e. u as a finite-valued function varying with x and t on a finite scale, approximately a function only of the combination x-t. To eliminate the explicit appearance of the small parameters, the evident procedure is to make a Galilean transformation to get rid of the u_x term, and then (because u is now slowly varying in the new time variable) rescale the time by factor ε. This produces the KdV equation; of course, it is merely the procedure of the first derivation recapitulated. If we do the same procedure on (29), the Galilean transformation which eliminates u_x reintroduces u_{xxx}:

$$u_t + \frac{\varepsilon}{4} uu_x - \frac{\alpha}{2} (u_{xxt} - u_{xxx}) = 0 \ . \tag{34}$$

Since now u is slowly varying in t, we drop the αu_{xxt} term as being of higher order, rescale the time, and again end up with the ubiquitous and inescapable KdV equation.

In short, the KdV equation really holds a unique and privileged place, and the equation proposed and studied by Benjamin et al. is merely an approximation to it. If one is willing, however, to countenance extra terms to improve the mathematical properties of the equation, there are other ways to do so. Thus R. Témam [1969] has studied the KdV equation (3) as the limit of the well posed higher order evolution equation

$$u_t + uu_x + u_{xxx} + \varepsilon u_{xxxx} = 0 \tag{35}$$

as ε approaches zero from above.

5. Conservation Laws

A (differential) conservation law says that a divergence vanishes. In our case (time and one space variable) this means it takes the form

$$T_t + X_x = 0 \ , \tag{36}$$

where T and X are functionals of u (and conceivably functions of t and x). Integrating over an interval gives

$$\frac{\partial}{\partial t} \int_a^b dx\ T + [X]_a^b = 0 \ , \tag{37}$$

whence T may be considered the density of a quantity which is conserved in the sense that the rate of change of the total amount of it in any interval [a,b] can be interpreted as the net effect due to the fluxes through the endpoints, each flux being -X evaluated locally. Conservation laws are of widespread occurrence and

frequently have both mathematical and physical significance.

The KdV equation is itself a conservation law, as seen by rewriting it in the form

$$(38) \qquad (u)_t + (\tfrac{1}{2} u^2 + u_{xx})_x = 0 .$$

However, it implies more conservation laws as well. Multiplying it by u gives what can be rewritten

$$(39) \qquad (\tfrac{1}{2} u^2)_t + (\tfrac{1}{3} u^3 - \tfrac{1}{2} u_x^2 + uu_{xx})_x = 0 .$$

A slightly more complicated manipulation [Whitham 1965] gives

$$(40) \qquad (\tfrac{1}{3} u^3 - u_x^2)_t + (\tfrac{1}{4}u^4 - 2uu_x^2 + u^2u_{xx} + u_{xx}^2 - 2u_x u_{xxx})_x = 0 .$$

Zabusky, Miura, and I found, by brute force calculations with undetermined coefficients, that this process could be continued indefinitely; as high a power as we started with (up to u^{10}, specifically), there was always a conservation law with that power occurring in the conserved density T, with T and X each being a polynomial in u and its x derivatives, and with each such polynomial having all terms of equal rank [the rank of a monomial $u^{a_0} u_x^{a_1} u_{xx}^{a_2} \ldots$ is defined as the number of factors $u, u_x, \ldots$ plus half the number of x-differentations, i.e. $a_0 + a_1 + a_2 + \ldots + \frac{1}{2}(a_1 + 2a_2 + 3a_3 + \ldots)$, this being the relevant combination because in the KdV equation uu_x combines with u_{xxx}, so that an extra factor u balances two extra x differentiations].

We further found that each such conservation law is essentially unique, viz., unique within two trivial transformations. The first is multiplication by a constant. The second is addition of an arbitrary x derivative to T and corresponding modification of X, for if $T_t + X_x = 0$ then also $(T+Q_x)_t + (X-Q_t)_x = 0$. Incidentally, we generally utilize this freedom, in effect the freedom to

integrate by parts, to put each polynomial conserved density into a canonical form such that each monomial in it has its highest order x derivative, if any, occurring at least squared. A conserved density which is a perfect x derivative is trivial, and usually of no significance.

Our group eventually even obtained an explicit formula for the n-th conservation law, assuming it existed, but were nevertheless unable to parlay this result straightforwardly into a proof (of existence) by induction. The breakthrough came from an inspired observation of Miura's.

Miura applied the same brute-force approach to the modified KdV equation (4), and found that it too possessed a seemingly endless sequence of conservation laws,

$$(v)_t + (\tfrac{1}{3} v^3 + v_{xx})_x = 0 , \tag{41}$$

$$(\tfrac{1}{2} v^2)_t + (\tfrac{1}{4} v^3 - \tfrac{1}{2} v_x^2 + v v_{xx})_x = 0 , \tag{42}$$

$$(\tfrac{1}{4} v^4 - \tfrac{3}{2} v_x^2)_t + (\tfrac{1}{6} v^6 - 3v^2 v_x^2 + v^3 v_{xx} + \tfrac{3}{2} v_{xx}^2 - 3 v_x v_{xxx})_x = 0 , \tag{43}$$

and so on. (To avoid giving a wrong impression, let me emphasize that this is a rare and remarkable property of a p.d.e. -- we have for instance proved that none of the still higher forms of the KdV equation, whether obtained from it by further raising the power of the dependent variable in the coefficient or by taking a derivative of higher order than the third, has more than three independent polynomial conserved densities. See [ ].)

Now Miura observed that the (slightly strange-seeming) transformation

$$u = v^2 + \sqrt{-6}\, v_x \tag{44}$$

takes conserved densities of the KdV dquation into those of the

modified KdV equation. (The fact that this transformation is complex, whereas up to now the equations and their solutions have been thought of as real, should not be allowed to disturb one. Think of it as purely formal at this point. When we change a coefficient by a negative factor shortly, everything will in fact become real.) Thus the conserved density u itself goes into v^2 (the term $\sqrt{-6}\ v_x$ being dropped as a perfect derivative), which is (twice) the second conserved density for the v equation. Similarly $\frac{1}{2}u^2 = \frac{1}{2}v^4 - 6\ v^2v_x - 3\ v_x^2$, which after dropping the perfect derivative term $6\ v^2v_x$ is (twice) the third conserved density for the v equation. This process continues all the way up. The omission of the first conserved density, v itself, in this process has a particular significance to be explained shortly.

In fact, not merely the conservation laws but the differential equations themselves are related by Miura's transformation. For it is easy to verify that if $u = v^2 + \sqrt{-6}\ v_x$, then

$$(45) \qquad u_t + uu_x + u_{xxx} = (2v + \sqrt{-6}\ \frac{\partial}{\partial x})\ (v_t + v^2v_x + v_{xxx})\ .$$

Thus if v satisfies the modified KdV equation, u satisfies the (original) KdV equation. Miura's transformation is very reminiscent of the famous transformation of Hopf [1950] and Cole [1951], which takes the quadratically nonlinear Burgers' equation

$$(46) \qquad u_t + uu_x - u_{xx} = 0$$

into the corresponding linear heat equation (2), except that it takes the quadratically nonlinear KdV equation into a similar but cubically nonlinear equation.

We now generalize Miura's transformation by introducing a parameter, without losing the quadraticity of the coefficient of v_x in the v equation, by replacing v everywhere by v+k, k a constant.

Then $u = (v+k)^2 + \sqrt{-6}\, v_x$ implies that $u_t + uu_x + u_{xxx}$
$= (2v + 2k + \sqrt{-6}\, \partial/\partial x)\ [v_t + (v^2 + 2kv + k^2)v_x + v_{xxx}]$.
To simplify the new v equation (obtained by setting the quantity in square brackets equal to zero) we make a Galilean transformation to get rid of the term $k^2 v_x$. With t and v unchanged, we replace $x \to x + k^2 t$, so that $\partial/\partial x$ is unchanged and $\partial/\partial t \to \partial/\partial t - k^2\, \partial/\partial x$. In order to keep the left side unchanged we replace $u \to u + k^2$, in effect exploiting the Galilean invariance of the KdV equation. This leads us to the result that if $u = v^2 + 2kv + \sqrt{-6}\, v_x$, then

$$u_t + uu_x + u_{xxx} = (2v + 2k + \sqrt{-6}\, \frac{\partial}{\partial x})\ [v_t + (v^2+2kv)v_x + v_{xxx}] .$$

We now choose to normalize the coefficient of the vv_x term (at the expense of the $v^2 v_x$ term) and so let $v = w/2k$. (Approaching our final result we want a new name for the much-transformed v.) To simplify the form of the coefficient of $\partial/\partial x$ in the resulting parenthetical operator we also set $k = \sqrt{6}/2\varepsilon$. The final result (essentially suggested by Clifford Gardner) is: If

$$u = w + i\varepsilon w_x + \frac{\varepsilon^2}{6} w^2 , \tag{47}$$

then

$$u_t + uu_x + u_{xxx} = (1+i\varepsilon \frac{\partial}{\partial x} + \frac{\varepsilon^2}{3} w)\ [w_t+(w+ \frac{\varepsilon^2}{6} w^2)w_x+ w_{xxx}] . \tag{48}$$

The reason for these successive transformations was to get the new variable, w, in a form which reduces to u in a limiting case, viz. $\varepsilon = 0$. It (or more precisely $\varepsilon w/\sqrt{6}$) reduces to v in the opposite limiting case $\varepsilon \to \infty$. (The reason for having chosen the coefficient of $\partial/\partial x$ to simplify is that it is the dominant one of the small terms as $\varepsilon \to 0$, since only ε^2 occurs elsewhere.)

Since w is merely a simply transformed version of v, the

"remodified" KdV equation

$$w_t + (w + \frac{\varepsilon^2}{6} w^2) w_x + w_{xxx} = 0 \quad , \tag{49}$$

like (4), has a sequence of conservation laws, which are obtainable from those for the KdV equation by the transformation (47). As before, there is an anomalous one not so obtained, viz. (49) itself rewritten:

$$(w)_t + (\frac{1}{2} w^2 + \frac{\varepsilon^2}{18} w^3 + w_{xx})_x = 0 \ . \tag{50}$$

There are now two crucial points. First, we note that (47), viewed in a formal way for $\varepsilon \to 0$, can be inverted to give w as a series of nonnegative powers of ε with coefficients which are polynomials in u and its derivatives, thus

$$\begin{aligned} w = u - i\varepsilon w_x - \varepsilon^2[u_{xx} + \tfrac{1}{6} u^2] + i\varepsilon^3[u_{xxx} + \tfrac{2}{3} uu_x] \\ + \varepsilon^4[u_{xxxx} + \tfrac{2}{3} uu_{xx} + \tfrac{5}{6} u_x^2 + \tfrac{1}{18} u^3] + \ldots \ . \end{aligned} \tag{51}$$

Second, since (49) is (formally) a consequence of the KdV equation, because the operator $1 + i\varepsilon \frac{\partial}{\partial x} + \frac{\varepsilon^2}{3} w$ in (48) can be formally inverted [its inverse is $1 - i\varepsilon \frac{\partial}{\partial x} - \varepsilon^2(\frac{1}{3} w + \frac{\partial^2}{\partial x^2}) + i\varepsilon^3(\frac{1}{3} w_x + \frac{2}{3} w \frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3}) + \ldots]$, and the KdV equation is independent of ε, it follows that order by order in ε, (50) provides a distinct conservation law for the KdV equation. Thus we construct an infinite sequence of conservation laws!

Actually, every second conservation law so obtained is trivial. To see this, we split w into real and imaginary parts, $w = r + is$, insert this into (47), and separate real and imaginary parts to obtain

$$u = r - \varepsilon s_x + \frac{\varepsilon^2}{6}(r^2 - s^2) , \tag{52}$$

$$0 = s + \varepsilon r_x + \frac{\varepsilon^2}{3} rs . \tag{53}$$

Solving the latter gives

$$\begin{aligned} s &= -\varepsilon r_x/(1+\frac{\varepsilon^2}{3} r) \\ &= [-\frac{3}{\varepsilon} \ln (1 + \frac{\varepsilon^2}{3} r)]_x , \end{aligned} \tag{54}$$

a perfect derivative. Hence the odd powers of ε, which, obviously, constitute collectively the imaginary part of w, are all trivial conserved densities.

Substituting (54) into (52) gives

$$u = r + \varepsilon^2 r_{xx}/(1+\frac{\varepsilon^2}{3} r) - \frac{\varepsilon^4}{2} r_x^2 (1+\frac{\varepsilon^2}{3} r)^2 + \frac{\varepsilon^2}{6} r^2 , \tag{55}$$

which generates r directly by inversion. To see that none of these conserved densities (the even powers of ε in w) is trivial, we note that the terms with u alone, without derivatives of u, which will be generated are just those generated by inverting (43) with r_x and r_{xx} set equal to zero, namely

$$u = r + \frac{\varepsilon^2}{6} r^2 . \tag{56}$$

But this quadratic equation has the solution

$$r = \frac{3}{\varepsilon^2} \left(-1 \pm \sqrt{1 + \frac{2}{3} \varepsilon^2 u}\right), \tag{57}$$

or, since the plus sign is clearly the appropriate one,

$$r = \frac{3}{\varepsilon^2} \sum_{n=1}^{\infty} \binom{1/2}{n} \left(\frac{2}{3} \varepsilon^2 u\right)^n . \tag{58}$$

Evidently every nonnegative even power of ε, and correspondingly every positive integral power of u, actually occurs in this expansion, since the binomial coefficient never vanishes. Because the

derivative of any polynomial in u, u_x, u_{xx}, ... obviously cannot contain a monomial which is a pure power of u, we see that there is a nontrivial (i.e. non-perfect-derivative) conserved density of each positive rank.

6. Schrödinger Equation

At this point it is convenient to eliminate the peculiar factor -6 by the replacements $u \to -6u$, $v \to \sqrt{-6}\, v$. Miura's transformation becomes: If

$$u = v^2 + v_x \,, \tag{59}$$

then

$$u_t - 6uu_x + u_{xxx} = (2v + \frac{\partial}{\partial x})\ (v_t - 6v^2 v_x + v_{xxx}) \,. \tag{60}$$

Accordingly we will take the KdV and modified KdV equations in the forms

$$u_t - 6uu_x + u_{xxx} = 0 \,, \tag{61}$$

$$v_t - 6v^2 v_x + v_{xxx} = 0 \,. \tag{62}$$

Now observe that Miura's transformation goes easily in only one direction. Given v we obtain u immediately from (59); and if v satisfies (62) then u is immediately seen to satisfy (61), by virtue of (60). The converse is less clear. Given u, we must solve the Riccati equation (59) for v, and the solution is not unique; and if u satisfies (61), it does not necessarily follow that v satisfies (62), since the operator $(2v + \partial/\partial x)$ in (60) has no inverse -- since it annihilates the function $\exp(-2\int^x v\,dx)$. [It, or rather its analog in w, $(1+i\varepsilon\,\partial/\partial x + \frac{\varepsilon^2}{3} w)$, as in (48), was usefully found invertible only because we were then working formally in series of nonnegative integral powers of ε. The function $\exp(\frac{i}{\varepsilon} x + \frac{i\varepsilon}{3}\int^x w\,dx)$ that is annihilated by this operator has

no such expansion.]

The handy way to solve the Riccati equation (59) for v, given u, is by means of the linearizing transformation

$$v = \frac{\psi_x}{\psi} , \tag{63}$$

which transforms (59) into

$$\psi_{xx} = u\psi . \tag{64}$$

Let us suppose the conceptually simplest boundary value problems for (61) and (62), that u and v are required to be periodic functions of x with a given period ℓ. (Obviously if v is periodic, u must be too, and with the same period.) It is natural to seek to have ψ similarly periodic. If v were known, ψ could be obtained from (63), as $\psi = \exp \int^x v\, dx$, and periodicity of ψ would require $\int_0^\ell v\, dx = 0$. It is tempting to hope to impose this as an extra condition to pick one special solution v out of the one-parameter family of solutions of (59). Unfortunately, however, this cannot generally be achieved, because (64) does not generally possess even a single periodic solution ψ (other than the unacceptable $\psi \equiv 0$); e.g., if $u > 0$ everywhere, then if $\psi \not\equiv 0$ were periodic it would have a positive maximum or a negative minimum, at neither of which (64) could hold.

It would actually be quite natural to abandon the imposition of periodicity on ψ, and instead, due to the homogeneity of its relations with u and v, merely require it to be "exponential-periodic" with period ℓ, that is to be the product of some exponential $\exp ax$ (a being constant) with a function of period ℓ. Such a ψ satisfying (64) can always be found, according to Floquet theory. It is perhaps fortunate that we thought of a different remedy first, since it opened up to us a vast new world.

That remedy was to utilize the Galilean invariance of the KdV equation to introduce a constant into Miura's transformation, much as done before for the proof of existence of infinitely many conservation laws. Because of the factor -6 introduced, if we make the replacement $u \to u - \lambda$ and wish to keep t and (61) invariant, we must replace $x \to x + 6\lambda t$.

We conclude from (59) and (60) that if

$$u - \lambda = v^2 + v_x \ , \tag{65}$$

then

$$u_t - 6uu_x + u_{xxx} = (2v + \frac{\partial}{\partial x}) \ [v_t - 6(\lambda+v^2)v_x + v_{xxx}] \ . \tag{66}$$

If we now linearize (65) as before by (63), we obtain

$$\psi_{xx} + (\lambda - u)\psi = 0 \ . \tag{67}$$

This is the well known so-called "time-independent Schrödinger equation," with u in the role of the potential. [The time referred to here, say τ, has nothing to do with the time t of our evolution equations but is represented by the residue of Fourier-analyzing out the quantum-mechanical time dependence, the term $\lambda\psi$, coming from an original term $i\psi_\tau$ in the time-dependent Schrödinger equation, cf. (18).] We shall here, as is customary, call it simply the "Schrödinger equation."

Now that we have the disposable parameter λ, we can choose it (given any periodic u) to permit a periodic solution ψ of (67); this is a standard Sturm-Liouville eigenvalue problem, and it is known that (67) possesses an infinite discrete spectrum of real eigenvalues $\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \lambda_5 < \lambda_6 \leq \ldots$ with $\lambda_n \to \infty$ as $n \to \infty$. Typically the λ_n are all distinct and each determines a real periodic eigenfunction ψ_n which is unique up to scaling (multiplication by a constant).

These last considerations were based on viewing (67) as an ordinary differential equation, since t enters only implicitly as a parameter and there are no differentiations with respect to it. In fact, however, u is supposed to evolve with time according to (61), hence the λ_n and ψ_n all evolve. To determine how they do so, we impose on v the "remodified" evolution equation

$$v_t - 6(\lambda + v^2)v_x + v_{xxx} = 0 \ , \tag{68}$$

where λ is some constant which is one of the eigenvalues λ_n of (67) at $t = 0$, say. It is important to note that we do not assume that λ remains an eigenvalue, since λ is taken to be a constant and the eigenvalues are presumed to vary with time. By (63) and (67) we have (65), hence (66), and are thus assured that u does evolve according to the KdV equation (61).

The evolution of ψ is easily obtained by first integrating (63) to obtain

$$\psi = \exp \int^x v \, dx \ . \tag{69}$$

We observe by the way that the condition of periodicity on ψ, namely $\int_0^\ell v \, dx = 0$, which is satisfied at $t = 0$ by virtue of the way λ was chosen (as one of the λ_n), remains true because v is a conserved density of (68) whose fluxes at the endpoints of the interval $[0,\ell]$ are equal, by the periodicity of v, and hence cancel in their contributions (one is inward, the other outward), so that the total amount in the one-period interval cannot change with time. Thus we know that ψ remains periodic. (Note that we have again made special use of the anomalous first conservation law of the v equation.) Differentiating (69) and using (68) gives

$$\psi_t = \psi \int^x v_t \, dx$$

$$= \psi \int^x [6(\lambda+v^2)v_x - v_{xxx}] \, dx$$

$$= \psi\big(6\lambda v + 2v^3 - v_{xx} + \text{const.}\big). \tag{70}$$

The constant of integration may be omitted since its presence merely corresponds to the freedom to rescale ψ arbitrarily at each time, i.e. to multiply ψ by an arbitrary function of t. (Alternatively, we could impose a normalization condition on ψ, such as $\int_0^\ell \psi^2 \, dx = 1$, and use this to determine the "constant" as a function of time.) Eliminating v by (63) and simplifying quickly gives for ψ the autonomous evolution equation

$$\psi_t - (6\lambda + 3\frac{\psi_{xx}}{\psi})\psi_x + \psi_{xxx} = 0 \, , \tag{71}$$

or, linear but not autonomous,

$$\psi_t - 3(\lambda + u)\psi_x + \psi_{xxx} = 0 \, . \tag{72}$$

To interpose a mathematical observation, it might be supposed that since this linear evolution equation has periodic coefficients, it would follow that an initially periodic ψ would necessarily evolve so as to remain periodic. This would obviate the relevance of the argument given just before to establish the periodicity of ψ. However, (71) does not determine the evolution of ψ in a local way, even though it is a local (i.e. differential) equation, because it is of "superparabolic" type. Three boundary conditions are undoubtedly needed to make the initial value problem well posed. Here the three conditions are taken to be $[\psi]_0^\ell = 0$, $[\psi_x]_0^\ell = 0$, $[\psi_{xx}]_0^\ell = 0$, in accord with the established periodicity of ψ.

To tie things up, we need merely observe that at any time t,

not merely at $t = 0$, we have (63) and (65), hence also (67). Thus the constant λ remains an eigenvalue of (67) for all time; or turning the statement around, every eigenvalue of (67) remains constant for all time! The argument presented here is the most intuitive and least calculational that I know, though it is a bit subtle in its logic.

Actually, of more real significance than the constancy of the eigenvalues of (67) under evolution of u in accord with (61), is the fact that ψ evolves according to a p.d.e. of its own, viz. (71). Under any deformation of the potential u in the Schrödinger equation one can obtain the corresponding deformation of ψ by standard perturbation theory, but this deformation is generally global in character, so that an integrodifferential equation for ψ would result. Only for very special deformations of u is the deformation of ψ expressible locally.

The systematic search for such special evolution equations for u was carried out by Lax [1968] and A. Lenard (see [ ]). The lowest order p.d.e. found is $u_t + u_x = 0$, which is trivial since it states that u is a function of only the single variable (x-t), and hence merely translates rigidly as t varies, whereupon ψ simply translates along with it. The next order p.d.e. found is the third order KdV equation (61). Then comes the fifth order p.d.e. (to be derived presently)

$$u_t + 3u^2u_x - 2u_xu_{xx} - uu_{xxx} + \frac{1}{10}u_{xxxxx} = 0 \ , \tag{73}$$

a seventh order one, and so on.

To see how this goes, we follow Lenard, approximately. Consider the equation

$$\psi_{xx} - h\psi = 0 \ , \tag{74}$$

where h and ψ are assumed to be periodic with period ℓ. Here h will later be replaced by $u-\lambda$, to make this into the Schrödinger equation, or by λr, to make this into a Sturm Liouville equation we shall also touch on.

We now perturb h and calculate the corresponding perturbation of ψ. Denoting perturbed quantities by a circumflex, we have

$$\hat{\psi}_{xx} - h\hat{\psi} - \hat{h}\psi = 0 \ . \tag{75}$$

This is an inhomogeneous linear equation for $\hat{\psi}$; it may be solved by quadratures by setting

$$\hat{\psi} = \alpha\psi \tag{76}$$

in it and using (74) to obtain

$$\alpha_{xx}\psi + 2\alpha_x\psi_x - \hat{h}\psi = 0 \ , \tag{77}$$

which is a first order equation for α_x. Multiplying by the integrating factor ψ leads to

$$\alpha_x\psi^2 = \int^x \hat{h}\ \psi^2\ dx \ . \tag{78}$$

Periodicity of α and hence $\hat{\psi}$ (assuming that of ψ, h, and $\hat{h}$) requires

$$\int_0^\ell \hat{h}\psi^2\ dx = 0 \ , \tag{79}$$

but this will be guaranteed anyway by what follows.

If α and hence $\hat{\psi}$ are to come out locally determined, $\hat{h}\psi^2$ must be integrable; we therefore set it equal to a perfect derivative, and since $\hat{h}\psi^2$ is of second degree in ψ and all our relations are homogeneous in ψ and its derivatives, we assume this to have the form

$$\hat{h}\psi^2 = (A\psi^2 + B\psi\psi_x + C\psi_x^2)_x \ . \tag{80}$$

Carrying out the indicated differentiations, eliminating ψ_{xx} by (74), and then separately equating the ψ^2, $\psi\psi_x$, and ψ_x^2 terms gives

$$\hat{h} = A_x + Bh \,, \tag{81}$$

$$0 = 2A + B_x + 2Ch \,, \tag{82}$$

$$0 = B + C_x \,. \tag{83}$$

Eliminating B by (83) and then A by (82) yields

$$\hat{h} = \frac{1}{2} C_{xxx} - 2C_x h - Ch_x \,. \tag{84}$$

We can now choose any function C and obtain $\hat{h}$ from this. We also obtain B and A from (83) and (82), so from (78) and (80),

$$\begin{aligned} \alpha_x &= A + \frac{B\psi_x}{\psi} + \frac{C\psi_x^2}{\psi} + \frac{k}{\psi^2} \\ &= \frac{1}{2} C_{xx} - Ch - \frac{C_x\psi_x}{\psi} + \frac{C\psi_x^2}{\psi^2} + \frac{k}{\psi^2} \,. \end{aligned} \tag{85}$$

If we choose the constant of integration $k = 0$, this becomes automatically integrable! Since $h = \psi_{xx}/\psi$, the integral is

$$\alpha = \frac{1}{2} C_x - \frac{C\psi_x}{\psi} \,, \tag{86}$$

where we have omitted the new constant of integration since its effect on $\hat{\psi} = \alpha\psi$ is a trivial manifestation in the perturbation of the freedom to scale the exact full ψ. Thus we conclude that

$$\hat{\psi} = \frac{1}{2} C_x\psi - C\psi_x \,. \tag{87}$$

As a first example, if we choose $C = -1$ (a multiplicative constant in C is unimportant, the constant being chosen merely for convenience) we obtain

$$\hat{h} = h_x \,, \quad \hat{\psi} = \psi_x \,. \tag{88}$$

Interpreting the perturbation as a time derivative, we get the

trivial case of (uniform) translation with time mentioned previously. A second example is provided by $C = -2\lambda h - 6\lambda^2$:

$$\hat{h} = -\lambda h_{xxx} + 6\lambda(h+\lambda)h_x \ , \tag{89}$$

$$\hat{\psi} = -\lambda h_x\psi + 2\lambda(h+3\lambda)\psi_x \ . \tag{90}$$

Thus we recover the KdV equation (set $h = u-\lambda$, $\hat{h} = \lambda u_t$) and (71), the evolution equation for ψ (set $\hat{\psi} = \lambda\psi_t$ and eliminate $h = u-\lambda = \psi_{xx}/\psi$). A third example, studied in recent work by Harry Dym and myself [unpublished], results from choosing $C = 2h^{-1/2}$ [to make the last two terms of (84) cancel], whereby

$$\hat{h} = (h^{-1/2})_{xxx} \ , \tag{91}$$

$$\hat{\psi} = (h^{-1/2})_x\psi - 2h^{-1/2}\psi_x \ . \tag{92}$$

With $h = \lambda r$, $\hat{h} = {}^{-1/2}r_t$, we obtain what I call Dym's equation, (7). (Higher order equations of Dym's type can also be obtained by the method about to be illustrated.)

Returning to the selection $h = u-\lambda$, we may ask what possibilities there are for C to make $\hat{h}$ given by (84) come out to be a function of u and its x-derivatives, to be interpreted as u_t. Actually, we can tolerate a power of λ as a common factor on the right, since it can be absorbed into t -- or $\hat{h}$ can be interpreted as u_t multiplied by a power of λ. Since λ occurs linearly on the right side of (84) (now that $h = u-\lambda$), it is natural to think of working with a finite series of integral powers of λ for C, chosen to cancel out power by power except at one end (only) of the series, where the term left "sticking out" can serve to provide the evolution equation. It is plausible to ask first for a formal infinite series for C which satisfies (84) with $\hat{h}$ omitted on the left. Such a series can then be truncated at any order to

provide an evolution equation.

There is a question whether the series should be in increasing or decreasing powers of λ, i.e. whether we should think of λ as small or large. If the former, we will have a third order equation to integrate at every step, and this seems impossible to carry out explicitly. In the latter case, however, we have merely a quadrature at every step; indeed, our equation can be written for recursive generation as

$$C = -\frac{1}{2}\lambda^{-1}\int^{x}[\tfrac{1}{2}C_{xxx} - 2C_x u - Cu_x]\,dx\ . \tag{93}$$

Starting with a constant (of integration) on the right, say 1, we obtain successively

$$C = 1 - \frac{1}{2}\lambda^{-1}\int^{x}[-u_x]\,dx + O(\lambda^{-2}) \tag{94}$$

$$= 1 + \frac{1}{2}\lambda^{-1}u + O(\lambda^{-2})\ ,$$

$$C = 1 + \frac{1}{2}\lambda^{-1}u - \frac{1}{4}\lambda^{-2}\int^{x}[\tfrac{1}{2}u_{xxx} - 3u_x u]\,dx + O(\lambda^{-3}) \tag{95}$$

$$= 1 + \frac{1}{2}\lambda^{-1}u - \frac{1}{8}\lambda^{-2}(u_{xx} - 3u^2) + O(\lambda^{-3})\ ,$$

and so on: to our surprise we are able to integrate explicitly at every step, apparently indefinitely. The reason for this, pointed out to me by Ira Bernstein in a different but remarkably similar situation, is that the homogeneous version of equation (84) for C, though linear, can be integrated explicitly by treating it nonlinearly; if we multiply by C we can integrate to obtain

$$\lambda = \frac{1}{2}CC_{xx} - \frac{1}{4}C_x^2 - C^2(u - \lambda)\ , \tag{96}$$

where the constant of integration (on the left) has been chosen so that $C = 1 + O(\lambda^{-1})$ in accord with the previous choice. Solving for C from the dominant term on the right, $C^2\lambda$, gives

(97) $$C = [1 - \lambda^{-1}(\tfrac{1}{2} CC_{xx} - \tfrac{1}{4} C_x^2 - C^2 u)]^{1/2} ,$$

which after formal expansion in nonpositive integral powers of λ is suitable for recursive generation of C. No further integration is required (they have already been performed all at once, as it were), and the result must evidently be identical with that obtained by the previous linear recursion -- which shows that at every step in the latter algorithm the integration will be explicitly possible.

Now if we truncate (95) at the first term, taking C = 1, (84) (now inhomogeneous) gives

(98) $$\hat{h} = - u_x ,$$

which with $\hat{h} = u_t$ is the simple translational evolution again. The next higher truncation is $C = 1 + \frac{1}{2} \lambda^{-1} u$, whence

(99) $$\hat{h} = \tfrac{1}{4} \lambda^{-1} u_{xxx} - \lambda^{-1} u_x (u-\lambda) - (1 + \tfrac{1}{2} \lambda^{-1} u) u_x$$

$$= \lambda^{-1} [\tfrac{1}{4} u_{xxx} - \tfrac{3}{2} u u_x] ,$$

which with $\hat{h} = - \frac{1}{4} \lambda^{-1} u_t$ gives the KdV equation (61). Note how the λ^0 terms (necessarily) cancelled. The next choice is $C = 1 + \frac{1}{2} \lambda^{-1} u - \frac{1}{8} \lambda^{-2} (u_{xx} - 3u^2)$, which produces (after automatic cancellation of the λ^0 and λ^{-1} terms)

(100) $$\hat{h} = \tfrac{5}{8} \lambda^{-2} [- \tfrac{1}{10} u_{xxxxx} + u u_{xxx} + 2 u_x u_{xx} - 3 u^2 u_x] ,$$

which leads to the evolution equation (73) given in advance.

The evolution equation for ψ, (72), is linear but contains λ; it is hardly surprising that all the eigenfunctions do not evolve according to a single common linear equation. But it is easy to obtain a common nonlinear equation: merely use (67) to eliminate λ in (72) to get

(101) $$\psi_t - (6u - 3 \frac{\psi_{xx}}{\psi}) \psi_x + \psi_{xxx} = 0 .$$

This is not very interesting, but becomes more so when multiplied by 2ψ and rewritten in the form

$$(\psi^2)_t - 6u(\psi^2)_x + (\psi^2)_{xxx} = 0 \ ; \tag{102}$$

the squares of the eigenfunctions do all evolve according to a single common linear equation! (This equation will play a significant role later.) Moreover, another nonsurprise, ψ^2 is a conserved density, though this is less apparent from (102) than from (71), which after multiplication by 2ψ gives

$$(\psi^2)_t + (-6\lambda\psi^2 - 4\psi_x^2 + 2\psi\psi_{xx})_x = 0 \ . \tag{103}$$

Of course this conservation law does not really give information about a solution u of the KdV equation, since ψ can be any periodic solution of (67) with eigenvalue λ, and there is no obvious independent way to set its scale (to normalize it).

This dfficulty may be obviated by the following consideration. Since the KdV equation (61) results from (99) by setting $\hat{h} = -\frac{1}{4}\lambda^{-1}u_t$, and $\hat{h} = (u-\lambda)\hat{} = \hat{u}$, we interpret $\hat{\psi}$ in (87) as $-\frac{1}{4}\lambda^{-1}\psi_t$, while $C = 1 + \frac{1}{2}\lambda^{-1}u$. [See lines above and below (99).] Both sides of (87) have the integrating factor ψ^{-3}, so that

$$(\psi^{-2})_t = (4\lambda C\psi^{-2})_x \ . \tag{104}$$

I.e. ψ^{-2} is also a conserved density. The constancy in time of $\left(\int_0^\ell \psi^2\,dx\right)\left(\int_0^\ell \psi^{-2}\,dx\right)$ is now genuinely informative, since the expression as a whole is invariant under rescaling of ψ. Both $\psi^2\int_0^\ell \psi^{-2}\,dx$ and $\psi^{-2}\int_0^\ell \psi^2\,dx$ are genuine conserved densities for the KdV equation, though not local functionals of u. (If ψ vanishes somewhere, $\int \psi^{-2}\,dx$ may be interpreted by taking the path of integration a little off the real axis, in case the functions involved are analytic, or equivalently but more generally by taking

the Hadamard finite part.)

Another way to generate the polynomial conserved densities (due to C. Gardner, and in fact the earliest way we knew), which is simplified by using the preceding results, is to solve (67) asymptotically for $\lambda \to \infty$ by the WKB method. We write (formally)

$$\psi = A \exp i \lambda^{1/2} \int^x B \, dx , \tag{105}$$

where A and B [which have nothing to do with A, B, C introduced in (80)] are series in nonnegative integral powers of $\lambda^{-1/2}$ (actually of λ^{-1}, it will turn out), the A being included for convenience, though it gives no greater generality (since it can be put logarithmically into the exponent and then differentially under the integral and thus absorbed into B). Substituting (105) into (67) gives

$$A_{xx} + 2A_x i\lambda^{1/2}B + A(-\lambda B^2 + i\lambda^{1/2}B_x + \lambda - u) = 0 . \tag{106}$$

We use the extra freedom provided by A to impose an extra condition, that the (formally, and actually if not yet so established) real and imaginary parts of (106) hold separately. The imaginary part gives $2A_xB + AB_x = 0$, or

$$B = A^{-2} \tag{107}$$

if we arbitrarily fix the amplitude of (105). With this the real part of (106) becomes $A_{xx} + A(-\lambda A^{-4} + \lambda - u) = 0$, from which A can be found recursively when it is solved for in the way appropriate for large λ,

$$A = [1 + \lambda^{-1}(\frac{A_{xx}}{A} - u)]^{-1/4} , \tag{108}$$

and the right side is expanded in powers of λ^{-1}. It is evident that the coefficients of these powers are polynomials in u and its derivatives.

Actually, it is not so much A we want as A^2, since that is a conserved density. The reason is that A obtained as just described is evidently real, hence so is B, whence the real and imaginary parts of ψ are $A \cos (\lambda^{1/2} \int^x B\, dx)$ and $A \sin (\lambda^{1/2} \int^x B\, dx)$. Since λ is (being taken to be) real, the real and imaginary parts of ψ separately satisfy (67), so that their squares are conserved densities. Adding, we see that A^2 is a conserved density. Clearly, the coefficient of each power of λ^{-1} in A^2 is by itself a conserved density, and is a polynomial in u and its derivatives.

Also A^{-2} is a conserved density. This is not so obvious from the fact that ψ^{-2} is, as one might have expected, but follows from (107). For, the periodicity condition on ψ as given by (105) is

$$\lambda^{1/2} \int_0^{\ell} B\, dx = 2\pi N \ , \tag{109}$$

where N is a (large) positive integer characterizing a pair of eigenfunctions (by being half the number of their zeros in the period ℓ). This shows that $\int_0^{\ell} B\, dx$ is constant, which implies (in the present context) that B is a conserved density. So, of course, is the coefficient of each power of λ^{-1} in its expansion. Naturally we do not obtain truly different polynomial conserved densities this way, but merely the same ones as before, modulo perfect derivatives.

7. Solitons

When the KdV equation is solved on the infinite interval $-\infty < x < \infty$ with essentially localized initial data, i.e. u(x,0) vanishing outside a finite interval or at least approaching zero sufficiently rapidly as $|x| \to \infty$, it is found that the asymptotic solution for $t \to \infty$ consists of a certain number (possibly zero)

of distinct widely spaced solitary waves travelling to the right at distinct velocities, as well as a dispersing "hash" left behind and decaying in amplitude. The solitary steady progressing wave solutions of (61) can be found by assuming for u the form

$$u(x,t) = U(x-ct) \ , \tag{110}$$

c being the velocity of the wave. This gives for U the ordinary differential equation

$$- cU' - 6UU' + U''' = 0 \ . \tag{111}$$

Integrating and assuming that $U \to 0$ as $|x| \to \infty$ gives

$$- cU - 3U^2 + U'' = 0 \ . \tag{112}$$

(In view of Galilean invariance this is not essentially more restrictive than assuming $U \to$ constant.) Multiplying by $2U'$ and again integrating gives

$$- cU^2 - 2U^3 + U'^2 = 0 \ . \tag{113}$$

An elementary quadrature yields finally

$$U = - \frac{c}{2} \operatorname{sech}^2 \frac{\sqrt{c}}{2} (x - ct - x_0) \ . \tag{114}$$

Thus a solitary wave can travel only to the right $(c > 0)$ -- for $c < 0$ (114) still makes sense, in that U is real, but it becomes oscillatory and fails to vanish at ∞ (is no longer solitary). It takes always the form of a dip, a decrease in u compared to the circumambient value but over only an essentially finite region, in view of the fast (exponential) decay of the hpyerbolic secant. (In the original form of the KdV equation, without the coefficient -6, the solitary waves are positive, increases in u.) We note that all solitary waves have the

same shape up to translation and scaling of the variables, and that the speed of the wave (c) is proportional to its amplitude (c/2) and inversely proportional to the square of its width ($2/\sqrt{c}$).

The remarkable fact is now that if we let the initial condition evolve backwards in time, we find for $t \to -\infty$ exactly the same number of solitary waves emerging from the hash, and with the same set of values of c. It is hard to resist identifying each of the solitary waves that appears at large positive t with its spitting image at large negative t. A somewhat metaphysical entity, which manifests itself as a solitary wave in the absence of disturbing influences (other waves or hash), may now be posited: this entity is called a soliton [Zabusky-Kruskal 1965].

If we eliminate the irrelevant emphasis on an initial state, on t = 0, we can describe the situation as follows. At large negative t let there be a number of solitary waves of distinct amplitudes and corresponding velocities travelling inward from infinity on the negative x-axis. During some essentially finite interval of time they interact, with each other and with whatever hash may be present. Eventually they emerge and separate, going off to infinity on the positive x-axis with their original shapes and velocities. During the interaction it is hard to know what "they" are -- there may be nothing present looking like the original waves, and if there are "waves" present they are certainly not solitary. We shall see that there is a very reasonable way to interpret the solitons during interaction and say just how they are distorted.

In the absence of hash we can obtain exact descriptions of the interactions of solitons by a simple variational procedure based on the conserved densities. For the KdV equation in the form (61), the corresponding constants of motion are

(115) $$I_1 = \int_{-\infty}^{\infty} u \, dx \ ,$$

(116) $$I_2 = \int_{-\infty}^{\infty} u^2 \, dx \ ,$$

(117) $$I_3 = \int_{-\infty}^{\infty} (u^3 + \tfrac{1}{2} u_x^2) \, dx \ ,$$

(118) $$I_4 = \int_{-\infty}^{\infty} (u^4 + 2uu_x^2 + \tfrac{1}{5} u_{xx}^2) \, dx \ ,$$

and so on. If we make I_3 stationary, for a prescribed value of I_2 (it is best to omit I_1, though we would include it if were were working on a finite interval with periodic functions, as earlier), we obtain the Euler variational equation

(119) $$2\alpha u + 3u^2 - u_{xx} = 0 \ ,$$

where α is a Lagrange multiplier. A function u satisfying (119) at one time continues to satisfy it as it evolves under the KdV equation -- for instance, if u minimizes I_3 for given I_2, it continues to do so because I_2 and I_3 are constants of motion. Thus we have found an ordinary differential equation (119), satisfied by a solution of the p.d.e. (61).

So far this isn't very interesting, because a solution of (119) is a soliton, as we see by comparison with (112). ($\alpha = \frac{1}{2}$ c.) But now let us do the same thing one level up. If we vary I_4 with I_2 and I_3 constrained to given values, we obtain

(120) $$2\beta u + \gamma(3u^2 - u_{xx}) + 4u^3 - 2u_x^2 - 4uu_{xx} + \tfrac{2}{5} u_{xxxx} = 0 \ .$$

Again, if (120) is satisfied at one time, it remains satisfied as u evolves. Now among the many solutions of (120) it is possible to find solitons. To see this one can use (119) to eliminate higher derivatives of u in favor of lower ones, $u_{xxxx} = (2\alpha u + 3u^2)_{xx}$, etc.

Eventually one will want also $u_x^2 = 2\alpha u^2 + 2u^3$ [cf. (113)]. The higher powers of u cancel automatically and (120) reduces to

$$(2\beta - 2\alpha\gamma + \frac{8}{5}\alpha^2)u = 0 , \tag{121}$$

whereupon the factor u may be omitted. For suitable values of β and γ this may have one or even two positive real solutions α. Suppose there are two. Then either one of two solitons satisfies (120). Hence so does a function consisting, loosely speaking, of both those solitons but "infinitely far apart." In the course of time the two solitons, moving at different velocities, will come closer together (if the faster is the one to the left) and eventually they will interact. At every time, however, (120) will remain satisfied and govern whatever shape their interaction takes.

Since we have an infinite sequence of conservation laws, the same argument can be carried out for any number of simultaneously interacting solitons. We obtain in each case a single higher order but still ordinary differential equation governing all stages of the interaction.

This is all very well, but still does not provide an interpretation of what the solitons are during the interaction. To that end we can apply a different variational argument. Let us minimize I_2 under the constraint that u permit n eigenfunctions $\psi^{(1)},\ldots,\psi^{(n)}$ with prescribed real eigenvalues $\lambda^{(1)},\ldots,\lambda^{(n)}$. The constraint is that a $\psi^{(m)}$ exist satisfying (67) for each $\lambda^{(m)}$,

$$\psi_{xx}^{(m)} + (\lambda^{(m)} - u)\psi^{(m)} = 0 , \quad m = 1,\ldots,n. \tag{122}$$

This is to be satisfied on the infinite interval, and we assume that the $\psi^{(m)}$ are square-integrable functions and normalize them,

$$\int_{-\infty}^{\infty} \psi^{(m)2}\, dx = 1 , \qquad m = 1,\ldots,n. \tag{123}$$

(We take the $\psi^{(m)}$ real; the $\lambda^{(m)}$ are evidently all negative.) With appropriate Lagrange multipliers $\alpha^{(m)}(x)$ and $\beta^{(m)}$, we require that

$$(124)\qquad \int_{-\infty}^{\infty} \left\{ \frac{1}{2} u^2 + \sum_{m=1}^{n} \left[\alpha^{(m)} \left(\psi_{xx}^{(m)} + (\lambda^{(m)} - u) \psi^{(m)} \right) + \beta^{(m)} \psi^{(m)2} \right] \right\} dx \ ,$$

be stationary with respect not only to variation of u but also each $\psi^{(m)}$. This gives the variational conditions

$$(125)\qquad u - \sum_{m} \alpha^{(m)} \psi^{(m)} = 0 \ ,$$

$$(126)\qquad \alpha_{xx}^{(m)} + \alpha^{(m)} (\lambda^{(m)} - u) + 2\beta^{(m)} \psi^{(m)} = 0 \ , \quad m=1,\ldots,n.$$

Multiplying the latter by $\psi^{(m)}$ and using (122) (after double integration by parts) and (123) gives

$$(127)\qquad 2\beta^{(m)} = 0 \ , \qquad m = 1,\ldots,n \ .$$

Therefore (126) becomes for $\alpha^{(m)}$ exactly the same equation as (122) is for $\psi^{(m)}$. We infer that $\alpha^{(m)}$ equals $\psi^{(m)}$, up to a constant multiplier:

$$(128)\qquad \alpha^{(m)}(x) = k^{(m)} \psi^{(m)}(x) \ .$$

(The second independent solution blows up as $|x| \to \infty$ in such a way as to make the terms in (124) not individually integrable, contrary to implicit assumption. If we were in the periodic case, the second independent solution would normally not be periodic and excluded on that ground.) From (125) we therefore obtain

$$(129)\qquad u = \sum_{m=1}^{n} k^{(m)} \psi^{(m)2} .$$

That is, the minimizing u is a linear superposition of the squares of its n eigenfunctions with the prescribed n eigenvalues.

As with the previous variational principles, since the conditions are constants of motion (specifically, here, I_2 and the $\lambda^{(m)}$), the conclusion continues to hold as u evolves. It is not a priori evident that the $k^{(m)}$ do not vary with t, but this in fact follows from (102); if we take the $k^{(m)}$ constant, the weighted sum of (102) applied to each $\psi^{(m)}$ gives back (61) and justifies the constancy of the $k^{(m)}$ a posteriori. Assuming that u separates into solitary waves for large $|t|$ (which is proved in [ ]), the $k^{(m)}$ can be evaluated as if $n = 1$ (since at most one term in the sum is nonnegligible at any one place). Integrating (129) over a limited but effectively infinite interval, with u having the single-soliton form (114), gives $-2\sqrt{c^{(m)}} = k^{(m)}$ by elementary quadrature. To relate $c^{(m)}$ to $\lambda^{(m)}$, a simple way is to observe that for $|x| \to \infty$, (112) becomes approximately $u'' \approx cu$ so that $u \sim \exp(-\sqrt{c}|x|)$, while (67) becomes $\psi_{xx} \approx -\lambda\psi$ so that $\psi \sim \exp(-\sqrt{-\lambda}\,|x|)$; since $u \sim \psi^{(m)2}$ here, we have $\sqrt{c} = 2\sqrt{-\lambda}$ or

$$k^{(m)} = -4\sqrt{-\lambda^{(m)}} \,. \tag{130}$$

The fact that $k^{(m)}\psi^{(m)2}$ is a solitary wave when it is the only nonnegligible term in the sum, and the validity of (129) throughout interaction, provide strong and almost compelling justification for viewing (129) as a decomposition of u as a linear superposition of solitons at all times, with the solitons of course distorted by interaction when they are not well separated. Indeed, it is plausible to generalize further and call $-4\sqrt{-\lambda^{(m)}}\,\psi^{(m)2}$ a soliton for any decently behaved u, not merely one made up of nothing but its solitons as given by (129). The right side of (129) then represents the soliton part of u, and the remainder provides an exact definition of the so-called hash even while it is interacting with the solitons.

As a final remark, it should be pointed out that a soliton

itself can be further analyzed into a superposition of poles in the complex plane. This is described briefly in [Kruskal 1974] and has been worked out in detail by W. R. Thickstun [to be published].

7. References

The main work of our group is distilled in a sequence of papers referred to in the text simply by the corresponding roman numeral. Those that have appeared by now are:

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