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Some Considerations on Closed Configurations of Magnetohydrostatic Equilibrium.

D. PALUMBO

EURATOM - Brussels

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1. – Toroidal confinement schemes of a hot plasma in a magnetic field are of increasing interest because of the obvious advantages which they have over open configurations. There are, however, difficulties which become immediately evident if one uses the guiding-center drift approximation for the description of the motion of the plasma particles.

In this approximation the drift of the guiding centers perpendicularly to the magnetic lines of force is determined by the field intensity gradient and by the curvature of the lines of force. The corresponding drift velocities $\boldsymbol{v}_g$, $\boldsymbol{v}_c$ [1], supposed small in comparison with the thermal velocities, are

$$\boldsymbol{v}_g = \frac{cw_\perp}{eB^2}\,\boldsymbol{t}\times\nabla B\,, \tag{1}$$

$$\boldsymbol{v}_c = \frac{2cw_\parallel}{eB\varrho}\,\boldsymbol{t}\times\boldsymbol{n} = \frac{2cw_\parallel}{eB\varrho}\,\boldsymbol{b}\,, \tag{2}$$

where $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ represent the unit vectors of the tangent, the principal normal, the binormal to the lines of force respectively, ϱ its curvature radius and $w_\parallel$, $w_\perp$ the kinetic energy in the instantaneous motion parallel and perpendicular to the magnetic field.

As was shown among others by BISHOP and SMITH [2] the effects of such drifts on the confinement are reduced, but not totally suppressed, by giving a rotational transform to the magnetic field.

Thus it seems interesting to consider the possibility of the magnetohydrostatic equilibrium for which the drift velocity $\boldsymbol{v}_d = \boldsymbol{v}_g + \boldsymbol{v}_c$ is tangent to the magnetic surface

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(1) *e.g.* D. V. SIVUKHIN: *Motion of charged particles etc.*, in *Rev. of Plasma Physics*, edited by LEONTOVICH, vol. 1 (New York, 1965); A. I. MOROZOV and L. S. SOLOV'EV: *Rev. of Plasma Physics*, edited by LEONTOVICH, vol. 2 (New York, 1966).

(2) A. S. BISHOP and C. G. SMITH: MATT 403, Plasma Physics Lab., Princeton University, 1966.

$P = \text{const}$. In the following we will consider only the case of an ideal plasma with scalar pressure P and zero electrical resistivity.

Apart from any consideration on experimental feasibility, these configurations have some interesting geometrical and physical properties; furthermore the solution in the particular case of the axial symmetry can be expressed in a simple analytic form.

2. – Taking into account the equilibrium equation which can be written

$$\nabla\left(4\pi P + \frac{1}{2}B^2\right) = \frac{1}{\varrho}B^2\boldsymbol{n} + B\frac{\partial B}{\partial s}\boldsymbol{t} \tag{3}$$

(s = the arc of the line of force), the condition $\boldsymbol{v}_d\cdot\nabla P = 0$ yields $\boldsymbol{b}\cdot\boldsymbol{N} = 0$, $\boldsymbol{N}$ being the normal to the magnetic surface. Together with the evident condition $\boldsymbol{t}\cdot\boldsymbol{N} = 0$ it implies that the lines of force are geodesics of the magnetic surface, and according to (3), it is equivalent to $\boldsymbol{b}\cdot\nabla B = 0$.

From this it follows—if there exists a dense set of magnetic surfaces on which the trajectories orthogonal to the lines of force cover ergodically the surface, and assuming spatial continuity of B—that $B^2 = \text{const}$ on each magnetic surface and in particular $\partial B/\partial s = 0$. In this case the coincidence between magnetic and drift surfaces was treated by MOROZOV and SOLOV'EV (1).

Similar results are reported by PFIRSCH and WOBIG (3) for the magnetic surface between a field-free plasma and a confining magnetic field.

It is easy to show that this class of configurations possesses in general several interesting properties, among others:

$\boldsymbol{j}$ being the current density, $\text{div}\,\boldsymbol{j}_\perp = 0$, thus $\text{div}\,\boldsymbol{j}_\parallel = 0$, from which, writing $\boldsymbol{j}_\parallel = \alpha\boldsymbol{B}$, $\partial\alpha/\partial s = 0$; if there exists a dense set of magnetic surfaces on which the lines of force cover ergodically the surface, $\alpha = \text{const}$ on each of them;

such configurations exhibit neither the phenomenon of local mirrors and trapped particles (4), nor their relative instabilities;

in the KADOMTSEV and POGUTSE (4) approximation the ion isothermal surface coincides with the magnetic surface;

in the case of nested toroidal surfaces with a magnetic axis, taking $\boldsymbol{N}$ directed outward with respect to the magnetic axis and indicating by $\mathrm{d}l$ the distance along $\boldsymbol{N}$ between two neighbouring magnetic surfaces, the equilibrium equation (3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}l}\left(4\pi P + \frac{1}{2}B^2\right) = -\frac{1}{\varrho}B^2. \tag{3'}$$

Thus along each line of force $\mathrm{d}l$ is proportional to ϱ and the center of curvature of the lines of force is on the side of the magnetic axis.

(3) D. PFIRSCH and H. WOBIG: *Plasma Physics and Controlled Nuclear Fusion*, in *Proc. of Culham Conference on Plasma Physics*, vol. 1 (Wien, 1966), p. 757.

(4) B. B. KADOMTSEV and P. POGUTSE: *Turbulent Processes in Toroidal Systems*, Kurchatov Institute Report IAE 1227, Culham translation 11, 1967.

3. – In the case of toroidal equilibrium configurations with axial symmetry, the condition $\boldsymbol{n}\|\boldsymbol{N}$ or its equivalent $\boldsymbol{b}\cdot\boldsymbol{\nabla}B=0$ implies (excluding the case of pure meridional magnetic field) $B^2=\text{const}$ on each magnetic surface. Assuming cylindrical co-ordinates R, θ, y (y axis of symmetry),

$$B_R=-\frac{1}{R}\frac{\partial\psi}{\partial y}\,,\quad B_\theta=\frac{T(\psi)}{R}\,,\quad B_y=\frac{1}{R}\frac{\partial\psi}{\partial R} \tag{4}$$

and the equilibrium equation ([5]) becomes ($x=R^2$)

$$4x\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+T\frac{\mathrm{d}T}{\mathrm{d}\psi}+4\pi x\frac{\mathrm{d}P}{\mathrm{d}\psi}=0\,. \tag{5}$$

The condition $B^2=B^2(\psi)$ yields

$$4x\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+T^2=xB^2(\psi)\,, \tag{6}$$

where $(1/R)\,\psi(x,y)$ is the azimutal component of the vector potential. At the same time $\psi(x,y)=\text{const}$ is one representation for the meridional cross-sections of the magnetic surfaces. Of course this family of curves can be represented by any $z(x,y)$ which is a function of ψ. If we choose z in such a way that $\mathrm{d}\psi/\mathrm{d}z=B$ from (5) and (6) we obtain the system

$$\begin{cases} 4xr+t=\dfrac{2}{\gamma}x-\dfrac{1}{2}\mu'\,,\\[2mm] 4xp^2+q^2=x-\mu\,, \end{cases} \tag{7}$$

where

$$p=\frac{\partial z}{\partial x}\,,\;\ldots,\; t=\frac{\partial^2 z}{\partial y^2}\,,\quad \mu=\frac{T^2}{B^2}\,,\quad \mu'=\frac{\mathrm{d}\mu}{\mathrm{d}z}\,,$$

and

$$\frac{1}{B^2}\frac{\mathrm{d}}{\mathrm{d}z}\left(4\pi P+\frac{1}{2}B^2\right)=-\frac{2}{\gamma}\,, \tag{7'}$$

μ, γ are of course functions of z only. The system (7) can be easily solved considering p, q which are functions of x, y as functions of x, $z(x,y)$ therefore

$$r=\left(\frac{\partial p}{\partial r}\right)_y=\left(\frac{\partial p}{\partial x}\right)_z+p\frac{\partial p}{\partial z}\,,\quad t=q\frac{\partial q}{\partial z}\,,$$

and the first of (7) reduces to $4x(\partial p/\partial x)_z=2x/\gamma$ and consequently

$$p=\frac{x}{2\gamma}+\beta(z)\,.$$

([5]) R. LÜST and A. SCHLÜTER: *Zeits. Naturfor.*, **12a**, 850 (1957).

Excluding the cases $1/\gamma = 0$, which gives for the magnetic surfaces a family of hyperboloids, and $q \equiv 0$, which corresponds to the cylindrical symmetry, both being open configurations, the system (7) can be written

$$(8) \qquad \begin{cases} p = \dfrac{1}{2\gamma}[x + \lambda(z)]\,, \\ 4xp^2 + q^2 = x - \mu\,, \end{cases}$$

which can be easily integrated starting from $dy = -(p/q)dx + (1/q)dz$. For a given value of z,

$$4\left(\frac{dy}{dx}\right)^2 = \frac{(x+\lambda)^2}{\gamma^2(x-\mu) - x(x+\lambda)^2}\,.$$

If $\lambda < 0$, the denominator can have a negative root x_1 and two positive roots $x_2 \leqslant x_3$; writing $x = x_3 - (x_3 - x_2)\sin^2\varphi$, $k^2 = (x_3 - x_2)/(x_3 - x_1) \geqslant 0$, $x_3 - x_1 = a$, the parametric equations for regular and closed meridional sections (and this condition determines λ and consequently x_1, x_2, x_3 as functions of k, a) are

$$(9) \qquad \begin{cases} R^2 = x = a\left(2\dfrac{\mathbf{E}}{\mathbf{K}} - 1 + k^2\cos^2\varphi\right) = b^2\mathbf{K}^2\left(2\dfrac{\mathbf{E}}{\mathbf{K}} - 1 + k^2\cos^2\varphi\right), \\ y = \sqrt{a}\left[E(\varphi, k) - \dfrac{\mathbf{E}}{\mathbf{K}}F(\varphi, k)\right] = b[\mathbf{K}E(\varphi, k) - \mathbf{E}F(\varphi, k)]\,, \end{cases}$$

where $a = b^2\mathbf{K}^2$. The standard notations for elliptic integrals, see *e.g.* [6], have been used.

The roots x_1, x_2, x_3 and λ being determined, γ, μ can easily be found from the expression of q^2 and also expressed as a function of k, a.

Finally the condition $\partial y/\partial z = 1/q$ gives $z = \frac{1}{3}\gamma$, and $b = b_0 = (2/\pi)R_0$, R_0 being the radius of the magnetic axis. In eq. (9) the magnetic surfaces are labelled by the parameter k or alternatively by γ, which is a monotonic function of k. The magnetic axis corresponds to $k = 0$ and it is important to note that for $k = 0$, $\gamma = 0$.

Thus the geometry of the magnetic surface and of the lines of force is fully determined (with the exception of a scale length factor R_0). In particular the azimutal increment θ_0 of the line of force during a complete turn around the magnetic axis is given by

$$\theta_0 = \frac{\pi\gamma}{R_0\sqrt{\mu}}\,\frac{1}{\mathbf{K}}\int_0^{\pi/2}\frac{\mu\,d\varphi}{x\Delta}\,.$$

The determination of P, B as functions of k or γ can be done starting from eq. (7′) which can easily be solved after giving some arbitrary relation $P = f(B)$.

It is clear that (7′) has a singularity on the magnetic axis. The most obvious physical solution near this axis can be obtained choosing

$$(10) \qquad 4\pi(P - P_0) + \tfrac{1}{2}B^2 = -mB^2, \qquad (m > 0)\,,$$

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[6] E. JAHNKE and F. EMDE: *Tabels of Functions*, Chap. 5 (New York, 1945).

which gives $B = A\gamma^{1/3m}$, $4\pi P = 4\pi P_0 - A^2(m+\frac{1}{2})\gamma^{2/3m}$, so that $B=0$ on the magnetic axis; this peculiarity makes the validity of the basic assumptions unreliable near the axis. For the configurations of the levitron type, the choice of $P = f(B)$ is much more arbitrary. It is worth noting that $j_{\parallel}/B$ is independent of $f(B)$.

More general equilibria having the same magnetic surfaces as given by eqs. (9) and satisfying eq. (5) but not (6) can be determined. The simplest example is obtained by adding an arbitrary constant to T^2.

Concerning the m.h.d. stability, of course the classical criteria can be expressed in a somewhat simplified form. In the case of the axial-symmetric configuration discussed above, the Mercier criterium [7] for stability against localized perturbation gives the simple condition

$$\frac{\Omega^2}{4\Gamma} + \left(\frac{5}{3} - \Omega\right)\frac{4\pi\gamma}{B^2}\frac{\mathrm{d}P}{\mathrm{d}\gamma} > 0\,,$$

where

$$\Omega = \frac{\sqrt{\mu}R_0}{\pi}\frac{\mathrm{d}\theta_0}{\mathrm{d}\gamma} \qquad \text{and} \qquad \Gamma = \frac{1}{\mathbf{K}}\int\limits_0^{\pi/2}\frac{\mu}{x-\mu}\frac{\mathrm{d}\varphi}{\Delta}\,.$$

Since on the magnetic axis $\Omega_0 = \frac{1}{3}$, $\Gamma_0 = \frac{1}{2}$ it is clear that the configurations defined by eq. (10) are unstable at least near the axis; this conclusion calls for the same remarks as expressed above.

The configuration of the levitron type can be stable against this class of perturbations for a sufficiently small pressure gradient.

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More details can be found in a forthcoming EURATOM report [8].

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(7) C. MERCIER: *Nuclear Fusion*, 1, 47 (1960).
(8) EUR-3748 *i* (Bruxelles, 1968).