Stability of a Linear Pinch

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It is now well known that a trapped longitudinal magnetic field has a stabilizing influence on a linear pinch. Once such a stabilized pinch is set up, however, diffusion will lead to mixing of the initially crossed fields; the torsion of the field lines will diminish and the plasma may ultimately become unstable. It is the purpose of this paper to study continuous plasma and field distributions in order to see at what point instability might be expected.

A variational principle has been given [1] which applies very nicely to the problem at hand. Briefly, one subjects the plasma to a displacement $\boldsymbol{\xi}$ and calculates the resulting change in the total energy, δW, of the hydromagnetic system. Stability then hinges on whether or not some $\boldsymbol{\xi}$ can produce a diminution of the energy. If we define the vector $\mathbf{Q} = \nabla\times[\boldsymbol{\xi}\times\mathbf{B}]$ it turns out that the change in energy is

$$\delta W = \int d^3x \{ \mathbf{Q}\cdot\mathbf{Q} - 4\pi\mathbf{J}\cdot[\mathbf{Q}\times\boldsymbol{\xi}] + \gamma p(\nabla\cdot\boldsymbol{\xi})^2 + (\nabla\cdot\boldsymbol{\xi})(\boldsymbol{\xi}\cdot\nabla p) \quad (1)$$

where γ is the specific heat ratio and p is the pressure. The integration is taken over the complete volume. If a displacement exists which makes δW negative, we have instability. In the linear pinch we are dealing with axial symmetry and the components of the magnetic field in cylindrical coordinates are $\mathbf{B} = (0, B_\theta, B_z)$. Moreover, it is assumed that B_θ, B_z and p are functions of r alone. It is then possible to analyze $\boldsymbol{\xi}$ in terms of displacements of the form

$$\boldsymbol{\xi} = [\xi_r(r), \xi_\theta(r), \xi_z(r)] \exp i(kz + m\theta). \quad (2)$$

The integration with respect to θ and z can be carried out and δW can be minimized with respect to ξ_θ and ξ_z by purely algebraic means. When this computation has been carried out we find

$$\delta W = \int_{r_1}^{r_2} \left\{ \frac{(rf\xi' + g\xi)}{m^2 + (kr)^2} + (f^2 - h)\xi^2 \right\} 2rdr, \quad (3)$$

where we have set

$$f \equiv kB_z + mB_\theta/r$$
$$g \equiv kB_z - mB_\theta/r$$
$$h \equiv (8\pi J_z/r)B_\theta$$
$$\xi \equiv d\xi'/dr,$$

and have dropped the subscript r on ξ for simplicity.

In order that δW be a minimum with respect to functions ξ, the displacement must be a solution of the Euler-Lagrange equation

$$\xi'' + P\xi' + Q\xi = 0\,,$$

where

$$P \equiv 3/r + 2f'/f - 2k^2r/[m^2 + (kr)^2] \quad (6)$$
$$Q \equiv -[(kr)^2 + (m^2 - 1)]/r^2 - 2k^2g/f[m^2 + (kr)^2] - 8\pi k^2p'/rf^2\,.$$

When ξ is chosen to be a solution of Eq. (5), the integrand of Eq. (3) is a perfect differential and we have

$$\delta W = 2\left[\frac{r^2f\xi(rf\xi' + g\xi)}{m^2 + (kr)^2}\right]_{r_1}^{r_2}. \quad (7)$$

Equation (5) must be solved subject to certain boundary conditions which we take to be

$$\xi(0) \text{ is finite,}$$
$$\xi(R) = 0. \quad (8)$$

In choosing the second boundary condition we have placed a perfectly conducting wall at $r = R$ in order to benefit from its stabilizing influence.[2, 3, 4] However, it will turn out that the wall has no stabilizing effect on the modes we shall study.

Our previous experience with the theory of instabilities [5] leads us to be particularly wary of "fluted" displacements which interchange magnetic field lines without bending them. The bending of field lines requires energy while interchanging them does not. The purpose of the trapped axial field was to stabilize the plasma by twisting the field lines so that any arbitrary displacement will bend some of them. Nevertheless, some displacements will bend some field lines less than others and it seems reasonable to expect that those displacements which bend the lines the least will be the most dangerous.

Now the magnetic field lines describe a set of spirals with a pitch

$$\mu \equiv B_\theta/rB_z \quad (9)$$

which, in general, varies from layer to layer. The lines of the $\boldsymbol{\xi}$ field (2), on the other hand, describe a set of spirals whose pitch is constant. If these two sets of spirals should match over a finite region of

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space then a displacement is possible which does not bend magnetic field lines. If, on the other hand, μ' is not zero anywhere, then it is nevertheless always possible to choose k such that the two spiral systems match at a particular radius. When this happens, displacements are possible which bend magnetic field lines very little in the neighborhood of this point. Accordingly, we shall assume that the worst choice of k, m is such that

$$f = kB_z + mB_\theta/r \tag{10}$$

vanishes at some point in $(0, R)$. Let the point where $f = 0$ be denoted by $r = a$.

Now we note that $r = a$ is a regular singular point of Eq. (5). The theory of such singularities tells us that the solutions to Eq. (5) can be written in the form

$$\xi = (r - a)^\nu \times \text{Power series in } (r - a),$$

where ν is a root of the indicial equation

$$\nu^2 + \nu + M^2 = 0, \tag{11}$$

with

$$M^2 \equiv \frac{8\pi p'}{rB_z^2}\left(\frac{\mu}{\mu'}\right)^2\Bigg|_{r=a}.$$

Thus we have

$$\nu_{1,2} = \tfrac{1}{2}(1 \pm (1 - 4M^2)^{1/2}). \tag{12}$$

If the roots are real we have $4M^2 \leqslant 1$. Now the boundary conditions (8) determine ξ uniquely (except for a normalization factor) in $(0 - a)$ and in $(a - r)$. Therefore, on either side of $r = a$ there will be an admixture of the more singular of the two solutions; i.e.

$$\xi = (r - a)^{\nu_1}\{1 + \text{higher order terms in } (r - a)\}, \tag{13}$$

where ν_1 is given by Eq. (12) with the minus sign. For this choice of ξ the integral (3) diverges and ξ is an improper function.

We can, however, consider the following proper displacement: ξ is the (properly normalized) solution of Eq. (5) in the ranges $0 \leqslant r \leqslant a - \epsilon$ and $a + \epsilon \leqslant r \leqslant R$. ξ is constant for $a - \epsilon < r < a + \epsilon$. If ϵ is chosen so small that the first term of Eq. (13) dominates the power series, then this first term can be substituted into Eq. (1) and it follows that δW is always positive.

If the roots are complex, it is convenient to write Eq. (12) in the form

$$\nu_{1,2} = -\tfrac{1}{2}(1 \pm i\beta)$$
$$\beta \equiv (4M^2 - 1)^{1/2}.$$

In terms of β, the displacement ξ is given by

$$\xi = |r - a|^{-1/2}\cos\{\tfrac{1}{2}\beta\log|r - a| + \phi\} \times [1 + \text{higher order terms}], \tag{15}$$

where ϕ is a constant phase angle determined by the boundary conditions. Again $\xi(r)$ is an improper function at $r = a$. We can circumvent this difficulty by choosing a displacement given by Eq. (15) over the range $0 \leqslant r \leqslant a - \epsilon$ and setting $\xi =$ constant for $a - \epsilon \leqslant r \leqslant a$. The range $a \leqslant r \leqslant R$ is treated in a similar fashion. The quantity ϵ is to be chosen so that the leading term in Eq. (15) dominates at $r = a - \epsilon$. When this choice of ξ is made we obtain

$$\int_0^a \left\{\frac{(rf\xi' + g\xi)^2}{m^2 + (kr)^2} + (f^2 - h)\xi\right\} 2r dr = \frac{a^3 B_z^2 \mu'^2}{2(1 + (a\mu)^2)}\Big[1 - 2M^2 + (1 - 2M^2)\cos 2\psi + \beta \sin 2\psi\Big], \tag{16}$$

where ψ is defined by

$$\psi \equiv \tfrac{1}{2}\beta \log \epsilon + \phi \tag{17}$$

and can be made anything we please (modulo 2π) by a suitable choice of ϵ. But the bracketed expression on the righthand side of Eq. (16) oscillates between the values 1 and $(1 - 4M^2)$ as ψ varies. If $(1 - 4M^2)$ is negative, it is possible to choose ϵ so that (16) is negative. Similarly, the integral taken from a to R can be made negative and we have found that complex roots imply instability.

The result of our investigation can be stated as a theorem:

A necessary condition that the $m \neq 0$ modes of a linear pinch be stable is that

$$(r/4)\,(\mu'/\mu)^2 + 8\pi p'/B_z^2 \geqslant 0 \tag{18}$$

at every point in the plasma.

The method by which we obtained this theorem from the Euler-Lagrange equation suggests the likelihood that the above inequality might also be a sufficient condition for stability. However, there are two major difficulties which will be discussed in the following paragraphs.

The foregoing analysis, leading to our theorem, suggests the importance of extremely localized mixing at any point of instability since the unstable modes we have found are those for which the radial displacement ξ is very small except in the immediate neighborhood of $r = a$, where the B and ξ fields interlace. Therefore, it is of interest to inquire: suppose a small region is unstable, in the sense that the inequality stated in our theorem is violated, what then happens?

The answer to such a question is very difficult to give, but an estimate has been made in the following manner: A displacement $\boldsymbol{\xi}$ of the unstable type is chosen. This leads to new values for p, B_z, B_θ and ρ in the neighborhood of $r = a$. Mixing is now simulated by replacing p, ρ, B_θ, B_z by the values obtained by averaging over θ, and we ask whether the new distribution is more or less stable using the above theorem as a criteria.

The result seems to go qualitatively as follows: The mixing of a small unstable region leads to a distribution which is less unstable on the inside and more unstable on the outside. Thus, if some interior shell were unstable, it would mix until stable and this, in turn, would upset the stability of the next shell which would proceed to mix and so on. In this fashion such an instability would propagate towards the surface. If, however, a layer near the surface is given excess stability, the outward progression of the mixing should be stopped. This excess stability of the surface layers ought to be insured if the B_z field were so programmed that μ' is made quite large in this region. The simplest programming appears to be one which would reverse B_z in the vacuum after the plasma has pinched.

The inequality appearing in our theorem is a necessary condition for stability. It might also be argued that this is a sufficient condition since it was obtained from the Euler-Lagrange equation of (3). There are, however, two difficulties to be overcome before this can be asserted. The first is that we did not truly minimize δW with respect to k, but rather used a heuristic *principle of minimum bending*. However, the principle seems physically reasonable and the objection does not seem to be very serious. The second difficulty is quite deep and arises because the Euler-Lagrange equation is a necessary condition, but in no way guarantees a minimum.

Some progress has been made in clarifying the second difficulty. Note that the condition that $\xi(0)$ be finite determines a solution of the Euler-Lagrange equation, and that $\xi(R) = 0$ determines another solution. It has been possible to prove the following theorem:

The necessary and sufficient condition for stability is that neither of the above-mentioned solutions to the Euler-Lagrange equation has a zero in the open interval $(0, R)$.

This makes further progress possible by application of the Sturmian theory to Eq. (5).

REFERENCES

1. I. B. Bernstein, E. A. Frieman, M. D. Kruskal and R. M. Kulsrud, U.S. Atomic Energy Comm., Report N.Y.O. 7315.
2. F. D. Shafranov, Atomnaya Energiya, *5*, 709 (1956).
3. R. J. Tayler, Proc. Phys. Soc. (London), *B70*, 31 (1957).
4. M. N. Rosenbluth, Proceedings of the Venice Conference on Ionization Processes (1957) (also Los Alamos Scientific Laboratory Report LA-2030).
5. M. D. Kruskal and J. L. Tuck, Proc. Royal Soc., to be published (also Los Alamos Scientific Laboratory Report LA-1716).