SOVIET PHYSICS JETP VOLUME 16, NUMBER 2 FEBRUARY, 1963

ON THE NONLINEAR THEORY OF THE DAMPING OF PLASMA WAVES

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Submitted to JETP editor January 14, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) 43, 490-499 (August, 1962)

The problem treated is that of the absorption of monochromatic plasma waves of finite (but small) amplitude in a rarefied plasma. The nonlinear distortion of the electron distribution function under the action of the field is obtained. The damping decrement is calculated, and its dependence on the amplitude Φ_0 of the wave is found to be $\Phi_0^{-3/2}$.

1. INTRODUCTION

THE damping of longitudinal waves in a plasma without collisions was first derived by Landau.[1] The cause of this damping is the strong interaction between the wave and those particles whose velocities are sufficiently close to the phase velocity of the wave (these particles are hereafter called resonance particles). Obviously the region of resonance particles is given in order of magnitude by the inequality

$$v_f - \Delta v \lesssim v \lesssim v_f + \Delta v, \qquad \Delta v \sim (2e\Phi_0/m)^{1/2}, \qquad (1)$$

where Φ_0 is the potential-field amplitude of the wave and $v_f = \omega/k$ is its phase velocity.

The Landau damping decrement is found from the linearized kinetic equation without the collision integral and is given by

$$\gamma_L = -\frac{\pi\omega_0}{2n} v_f^2 \left.\frac{df_0}{dv}\right|_{v=v_f} = \sqrt{\frac{\pi}{8}}\,\omega_0 \left(\frac{v_f}{v_T}\right)^2 \exp\left(-\frac{v_f^2}{2v_T^2}\right), \qquad (2)$$

where ω_0 is the Langmuir frequency, n is the density of the plasma, $v_T = (T/m)^{1/2}$ is the thermal velocity of the electrons, and $f_0(v)$ is the distribution function of the unperturbed state of the plasma with the temperature T (the "background" distribution function). Here no account is taken of the inverse influence of the plasma waves on the "background."

Recently there has been developed a so-called quasi-linear theory of plasma oscillations[2,3] which makes it possible to take into account in first approximation the distortion of the "background" under the influence of the oscillations. The main manifestation of this distortion is the formation of a "plateau" (when collisions are neglected) on the background distribution function in the region of the resonance particles. On the other hand the collisions tend to make the distribution function Maxwellian, and the final form of the function is determined by the competition of these processes (see Fig. 1). In particular, the linearized theory of Landau, which does not take into account the nonlinear distortion of the background, is valid in the case in which the collisions succeed in making the distribution function completely Maxwellian, that is, for sufficiently small oscillations. The region of applicability of the quasi-linear theory is also confined to small waves, but here the energy of the waves must be small in comparison with the thermal energy, whereas in the linear theory the energy of the waves must be much smaller than the product of the thermal energy and a coefficient which depends on the collision frequency and is very small in comparison with unity[2] (see also the footnote in the present paper).

There is one more condition limiting the region of applicability of the quasi-linear theory: this theory assumes that the average background is homogeneous in space, and therefore can be applied only to wave packets that are sufficiently broad (in k-space), and in particular does not apply at all to plane waves, for which the spatial periodicity of the background distribution function in the resonance region is important.

The present paper deals with this second limiting case, namely the nonlinear interaction of a plasma with a monochromatic Langmuir wave, and with the consequent damping of the wave. Here, just as in the quasi-linear theory, we shall confine ourselves to the treatment of waves that are small in the sense that

$$e\Phi/T \ll 1, \qquad (3)$$

where Φ is the potential field of the wave and T is the temperature of the plasma.

2. THE FUNDAMENTAL EQUATION

For small waves the absorption coefficient is determined by the shape of the distribution function in the resonance region (1).[2,3] Therefore it is just in the region (1) that we shall be interested in the nonlinear distortion of f(v). Outside the resonance region, as in [2,3], we can neglect the deviation of the distribution function from the Maxwellian form (see Fig. 1). This allows us to assume that the phase velocity is defined in just the same way as in the linear theory, namely

$$\left(\frac{\omega}{k}\right)^2 = \left(\frac{\omega_0}{k}\right)^2 + 3\langle v^2\rangle_0, \qquad \langle v^2\rangle_0 = \int v^2 f_0(v)\, d^3v. \quad (4)$$

In the determination of the distribution function in the resonance region it is essential to take the collisions into account. Complete neglect of the collisions would mean that there would be rapidly established a state of the plasma (of the type of the "plateau" in Fig. 1) in which there would cease to be any damping.[2-4] Thus we cannot neglect the collision integral in the kinetic equation.

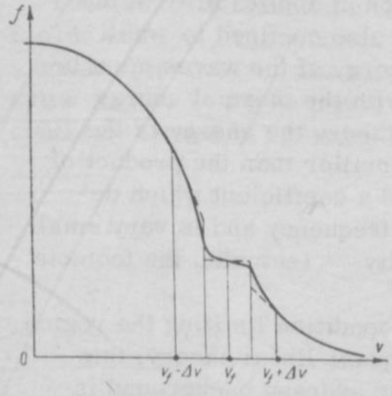

FIG. 1. Shape of the distribution function when the inverse action of the wave on the "background" is taken into account. Outside the resonance region the distribution function is Maxwellian.

The importance of including the collisions in the resonance region even when the collision frequency is very small can be understood easily from the following arguments. The time Δt for a change of the velocity by the amount of the width of the resonance region owing to Coulomb repulsions (with scattering through small angles) is given in order of magnitude by $\Delta t \sim (\Delta v)^2/D$, where D is the diffusion coefficient in the velocity space: $D \sim v_T^2/\tau_c$. Here τ_c is the effective relaxation time (cf. e.g., [5]), and Δv is given by Eq. (1). Thus

$$\Delta t \sim \tau_c\,(e\Phi/T). \quad (5)$$

For sufficiently small values of $e\Phi/T$ this time is very small, and in particular much smaller than the time of damping of the wave, which is given by Eq. (42).

Thus we write the kinetic equation in the form

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} - \frac{e}{m}\frac{\partial \Phi}{\partial \mathbf{r}}\frac{\partial f}{\partial \mathbf{v}} = -\,\mathrm{St}\,(f), \quad (6)$$

where St(f) is the collision integral, written in the Landau form:

$$\mathrm{St}\,(f) = \frac{2\pi e^4 L}{m^2}\frac{\partial}{\partial v_i}\int\left(f\,\frac{\partial f'}{\partial v'_k} - f'\,\frac{\partial f}{\partial v_k}\right)\frac{u^2\delta_{ik} - u_i u_k}{u^3}\,d^3u; \quad (7)$$

$u_i = v_i - v'_i$. As in [2], we can linearize the right member of Eq. (7) by replacing $f' = f(v')$ by $f_0(v')$, the Maxwellian distribution function, because we are taking the deviation of $f(v')$ from $f_0(v')$ into account only in the resonance region, and the contribution to the integral from this region can be neglected because it is small. Thus the linearized collision integral is obtained in the form[2]

$$\mathrm{St}\,(f) = -\frac{L\omega_0^4}{4\pi n}\frac{\partial}{\partial v_i} \times \left\{\frac{1}{v^3}\left[v_i f + \left(v^2\delta_{ik} - v_i v_k - \frac{T}{m}\frac{v^2\delta_{ik} - 3v_i v_k}{2v^2}\right)\frac{\partial f}{\partial v_k}\right]\right\}, \quad (8)$$

where L is the Coulomb logarithm.

Substituting Eq. (8) in Eq. (6) and integrating both members over v_y and v_z, we get (neglecting terms that are small for $v \sim v_f \gg v_T$, and also the contribution from the resonance region)

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{d\Phi}{dx}\frac{\partial f}{\partial v} = \frac{3}{2\tau_D}\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}v_T^2 + vf\right). \quad (9)$$

The quantity τ_D is defined by the formula

$$\tau_D = m^2 v_f^3/8\pi e^4 nL \quad (10)$$

and has the meaning of the effective collision time for particles in the resonance region [cf. [5], Eq. (5.22)].

We note that the term in the right member of Eq. (9) which describes the collisions has been used earlier by Lenard and Bernstein.[6] These authors justified its introduction by the fact that it is the simplest expression satisfying the dimensional requirements which preserves the main features of the Landau collision integral—namely, is of the form of a divergence and describes a diffusion in velocity space analogous to Fokker-Planck diffusion; it becomes zero when the Maxwell distribution is substituted in it. We see that this term can be obtained rigorously from the exact collision integral of Landau for a sufficiently narrow one-dimensional region in velocity space far from the thermal velocity. (The width of this region is determined by the inequalities $\Delta v \ll v_f - v_T$, $v_f \gg v_T$.) At the same time we have obtained the exact value of the coefficient of the collision integral. The value of this coefficient determined in [6] is not quite exact [it differs from our value by a factor $3(v_f/v_T)^3$].

Equation (9) becomes much simpler if we go over to the rest system of the wave and note that owing to the smallness of the damping of the wave we can neglect the time-derivative term in this system (the condition for the applicability of this approximation will be given at the end of this paper). Thus in the rest system of the wave the kinetic equation takes the form

$$v\frac{\partial f}{\partial x} + \frac{e}{m}\Phi'(x)\frac{\partial f}{\partial v} = \frac{3}{2\tau_D}\frac{\partial}{\partial v}\left[\frac{\partial f}{\partial v}v_T^2 + (v + v_f)f\right], \quad (11)$$

where v now means the velocity in the new reference system, and the resonance region is defined by the condition

$$|v| \sim (2e\Phi_0/m)^{1/2}. \quad (11a)$$

We write the equation of the wave in the system in which it is at rest in the form

$$E = E_0 \sin kx,$$

$$\Phi = \Phi_0(1 - \cos kx)/2 = \Phi_0 \sin^2(kx/2). \quad (12)$$

We have chosen the arbitrary constant in the potential so that the potential vanishes at $x = 0$. It is helpful to change to the dimensionless variables

$$v/v_T = u, \quad e\Phi/T = \varphi, \quad kx = y, \quad v_f/v_T = \alpha, \quad (13)$$

and when we do this the kinetic equation (11) takes the form

$$u\frac{\partial f}{\partial y} + \varphi'(y)\frac{\partial f}{\partial u} = \mu\frac{\partial}{\partial u}\left[\frac{\partial f}{\partial u} + (u + \alpha)f\right], \quad (14)$$

where $\varphi = \varphi_0 \sin^2(y/2)$, and

$$\mu = 3/2\, kv_T\tau_D. \quad (15)$$

It can be seen from Eq. (15) that for sufficiently small collision frequencies ($\tau_D^{-1} \ll kv_T$) we can regard the parameter μ as small. Besides this, Eq. (14) contains another small parameter—the dimensionless wave amplitude $\varphi_0 = e\Phi_0/T$. The solutions of Eq. (14) in the resonance region are quite different for different values of the ratio of μ and φ_0.

For $\varphi_0 \ll \mu$ (weak waves) we can look for a solution of Eq. (14) in the form of a power series in φ_0, and we get the usual linear theory, which leads to the Landau damping (2). We are here interested in the opposite case of strong waves with

$$\mu \ll \varphi_0, \quad (16)$$

for which there is a strong distortion of the distribution function in the resonance region and the damping differs in principle from that found by Landau.

It is convenient to rewrite Eq. (14) in new variables y, ϵ, where ϵ is the dimensionless total energy of an electron and y is the previous dimensionless coordinate:

$$y = kx, \quad \varepsilon = (mv^2/2 + e\Phi)/T = u^2/2 + \varphi. \quad (17)$$

In these variables Eq. (14) takes the form

$$\frac{\partial f}{\partial y} = \nu\frac{\partial}{\partial \varepsilon}\left[\pm\sqrt{\varepsilon - \varphi(y)}\left(f + \frac{\partial f}{\partial \varepsilon}\right) + cf\right], \quad (18)$$

where we have used the notations

$$c = \alpha/\sqrt{2} = v_f/v_T\sqrt{2}, \quad \nu = \sqrt{2}\mu = 3/\sqrt{2}kv_T\tau_D. \quad (19)$$

We can regard c as the dimensionless phase velocity of the wave and ν as the dimensionless collision frequency. The two signs of the square root in Eq. (18) correspond to the two possible directions of the velocity of the particles relative to the wave for a prescribed energy; thus Eq. (18) breaks up into two equations which correspond to the two possible directions of the velocity of the particles relative to the wave.

3. THE DISTRIBUTION FUNCTION IN THE OUTER AND INNER REGIONS

We must look for the solution of Eq. (18) separately for two regions—the outer ($\epsilon > \varphi_0$) and the inner region ($\epsilon < \varphi_0$) (Fig. 2). The inner region

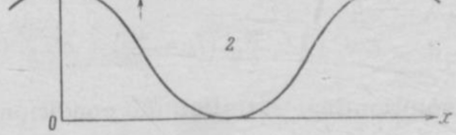

FIG. 2. The distribution function in the outer and inner regions: 1 – outer region, 2 – inner region, 3 – boundary layer

corresponds to particles captured by the wave and moving along with it. In accordance with Eq. (16) we can treat ν as a small parameter and look for the solution as a power series in ν:

$$f(y, \varepsilon) = f_0(\varepsilon) + \nu f_1(y, \varepsilon) + \ldots, \quad (20)$$

where $f_0(\epsilon)$ does not depend on y, since it is the solution of Eq. (18) for $\nu = 0$. Furthermore it is necessary to note that the small parameter ν occurs in Eq. (18) with the higher derivative with respect to ϵ. Therefore the expansion (20) will not describe the true solution in a narrow boundary layer lying between the outer and inner regions. The width of this boundary layer is $\sim \nu$, i.e., much smaller than the width of the resonance region (c.f. e.g., [7], where the phenomena in the boundary layer are analyzed for equations with a small parameter associated with a higher derivative). The contribution of the boundary layer to the damping of the wave will be examined in Sec. 4.

Let us find the solution of Eq. (18) in the outer region. Substituting Eq. (20) in Eq. (18), we get

$$\frac{\partial f_1}{\partial y} = \frac{\partial}{\partial \varepsilon}\left[\pm \sqrt{\varepsilon - \varphi(y)}\left(f_0 + \frac{\partial f_0}{\partial \varepsilon}\right) + cf_0\right]. \quad (21)$$

Owing to the periodicity of the field of the wave the distribution function $f(y, \epsilon)$ must also be periodic in y (with period 2π). Therefore when both members of Eq. (21) are averaged over y the left member must vanish, and we get

$$\frac{d}{d\varepsilon}\left[J(\varepsilon)\left(f_0 + \frac{df_0}{d\varepsilon}\right) + cf\right] = 0;$$

$$J(\varepsilon) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{\varepsilon - \varphi_0 \sin^2(y/2)}\, dy = \frac{2}{\pi}\sqrt{\varepsilon}\, E(k). \quad (22)$$

Here $E(k)$ is the complete elliptic integral of the second kind with modulus $k = (\varphi_0/\epsilon)^{1/2}$.

It is obvious that for $\epsilon \to \infty$ the function $f_0(\epsilon)$ must approach asymptotically the distribution function for a Maxwellian plasma moving with the velocity $-v_f$, or, in dimensionless variables

$$f_0(\varepsilon) \to (n/\sqrt{2\pi}v_T)\exp\{-(\pm\sqrt{\varepsilon} + c)^2\}, \quad \varepsilon \to \infty, \quad (23)$$

where the signs of the square root have the same meaning as in Eq. (18). Integrating Eq. (22) with this boundary condition, we get the following first-order equation for $f_0(\epsilon)$:

$$\pm\frac{2}{\pi}\varepsilon^{1/2}E\left(\sqrt{\frac{\varphi_0}{\varepsilon}}\right)\left(f_0 + \frac{df_0}{d\varepsilon}\right) + cf_0 = 0,$$

and the solution that satisfies the condition (23) is

$$f_0^{\pm}(\varepsilon) = \frac{n}{\sqrt{2\pi}v_T}e^{-c^2}\exp\left[-\varepsilon \mp \frac{c\pi}{2}\varphi_0^{1/2}\int_1^{\varepsilon/\varphi_0}\frac{dt}{t^{1/2}E(t^{-1/2})}\right]. \quad (24)$$

It is easy to obtain asymptotic expressions for this function in the two limiting cases $\epsilon \sim \varphi_0$ and $\epsilon \gg \varphi_0$:

$$f_0^{\pm}(\varepsilon) = A\exp\left[-\varepsilon \mp \tfrac{1}{2}c\pi\varphi_0^{-1/2}(\varepsilon - \varphi_0)\right],$$
$$(\varepsilon - \varphi_0)/\varphi_0 \ll 1, \quad (24a)$$

$$f_0^{\pm}(\varepsilon) = A\exp[-\varepsilon \mp 2c\varepsilon^{1/2}], \quad \varepsilon \gg \varphi_0;$$
$$A = (n/\sqrt{2\pi}v_T)e^{-c^2}. \quad (24b)$$

The next term of the expansion (20) for the outer region can be obtained from Eq. (21) by an elementary integration, but in what follows we shall need only $\partial f_1/\partial y$, which is given directly by Eq. (21).

The determination of the distribution function $f(y, \epsilon)$ for the inner region is greatly simplified owing to the fact, which can easily be verified, that for $\epsilon < \varphi_0$ (i.e., for particles captured by the wave) $f(y, \epsilon)$ must be the same for particles moving in the positive and negative directions relative to the wave; that is, $f^+(\epsilon, y) = f^-(\epsilon, y)$ (see also [4]). This is possible only if the term in the right member of Eq. (18) that contains the radical is zero, i.e., $f + \partial f/\partial\epsilon = 0$. Solving this equation together with Eq. (18), we get

$$f(y, \varepsilon) = a\exp(-\varepsilon - \nu cy) = ae^{-\varepsilon}(1 - \nu cy + \ldots),$$
$$-2\arcsin\sqrt{\varepsilon} \leqslant y \leqslant 2\arcsin\sqrt{\varepsilon}, \quad (25)$$

where a is a constant.

For the determination of a we note that with our restrictions ($\varphi_0 \ll 1$) the distribution function in the resonance region must differ only slightly from the Maxwell distribution; the only important change is in the derivative (see Fig. 1). For this to be so we must have

$$a = \frac{n}{\sqrt{2\pi}v_T}\exp\left(-\frac{v_f^2}{2v_T^2}\right) + O(\varphi_0) \simeq \frac{n}{\sqrt{2\pi}v_T}e^{-c^2}. \quad (25a)$$

We call attention to the fact that apart from terms that vanish for $\varphi_0 \to 0$ the constant a is the same as the A in Eq. (24b).

Thus we can write the following expression for the distribution function in the inner region:

$$f(\varepsilon, y) = f_0(\varepsilon)e^{-\nu cy},$$
$$f_0(\varepsilon) = (ne^{-c^2}/\sqrt{2\pi}v_T)e^{-\varepsilon} + O(\varphi_0), \quad (26)$$

where $f_0(\epsilon)$ is the distribution function for the inner region in the zeroth approximation in ν. It is interesting to note that $f_0(\epsilon)$ has the form of the Maxwell-Boltzmann distribution (ϵ is the energy divided by T) and goes over continuously into $f_0(\epsilon)$ for the outer region, as found from the asymptotic formula (24a). Their derivatives at $\epsilon = \varphi_0$ are not equal, however (in this connection we recall the existence of a narrow boundary layer for $\epsilon \sim \varphi_0$, in which there is a continuous transition between the derivatives of the distribution functions in the inner and outer regions).

4. THE DAMPING DECREMENT OF THE WAVE

The damping decrement of the wave is determined by the quantity $\overline{\dot{W}}$ —the mean value of the time derivative of the energy density of the wave (to simplify the writing we shall hereafter omit the bar over $\dot{W}$). Obviously $\dot{W} = -\mathbf{E}\cdot\mathbf{j}$, where $\mathbf{E}$ is the field strength of the wave and $\mathbf{j}$ is the current density induced by it. In going over to the system in which the wave is at rest we must note that $\mathbf{j} = \mathbf{j}' + v_f\rho$, where $\mathbf{j}'$ is the current density in the rest system of the wave and ρ is the charge density. Therefore

$$\dot{W} = -\overline{(\mathbf{E}\mathbf{j})} = -v_f\,\overline{(\mathbf{E}\rho)}, \quad (27)$$

where the averaging is taken over the coordinate x in the system of the wave. Since in this system the distribution function can be regarded as stationary, we have $\partial j'/\partial x = 0$ (the equation of continuity), $j' = \text{const}$, and the first term in Eq. (27) vanishes. Consequently

$$\dot{W} = -v_f \overline{E\rho} = -\frac{\omega}{2\pi}\int_0^{\lambda}\frac{\partial\Phi}{dx}\rho\,dx, \quad \lambda = \frac{2\pi}{k}. \tag{28}$$

Inserting here the value

$$\rho = \int_{-\infty}^{\infty} f\,dv,$$

where $f(v, x)$ is the distribution function in the system of the wave, and going over to the new variables y, ϵ defined by the relations (17), we get

$$\dot{W} = \frac{\omega T v_T}{\sqrt{8\pi}}\int_{-\pi}^{\pi} dy \int_{\varphi(y)}^{\infty}\frac{d\varepsilon}{[\varepsilon-\varphi(y)]^{1/2}}\varphi'(y)(f^+ + f^-), \tag{29}$$

where $f^+(\epsilon, y)$ and $f^-(\epsilon, y)$ correspond to electrons with positive and negative velocities relative to the wave.

Let us break up the integral over ϵ in Eq. (29) into three parts corresponding to the contributions from the outer and inner regions and the boundary layer:

$$\dot{W} = \dot{W}^{(1)} + \dot{W}^{(2)} + \dot{W}^{(3)}. \tag{30}$$

1. Let us calculate the contribution from the outer region, confining ourselves to the terms of lowest order in φ_0. We write

$$\dot{W}_{\pm}^{(1)} = \frac{\omega T v_T}{\sqrt{8\pi}}\int_{-\pi}^{\pi} dy \int_{\varphi_0}^{\infty} d\varepsilon\,[\varepsilon-\varphi(y)]^{-1/2}\varphi'(y)\,f^{\pm}(\varepsilon, y), \tag{31}$$

$$\dot{W}^{(1)} = \dot{W}_{+}^{(1)} + \dot{W}_{-}^{(1)}. \tag{31a}$$

Changing the order of integrations in Eq. (31) and then integrating by parts, using the periodicity of the integrand in y and substituting $\partial f_1^{\pm}/\partial y$ from Eq. (21), we get (confining ourselves to terms of first order in the collision frequency)

$$\dot{W}_{\pm}^{(1)} = \frac{\omega T v_T}{\sqrt{8\pi}}\nu\int_{\varphi_0}^{\infty} d\varepsilon\int_{-\pi}^{\pi} dy\sqrt{\varepsilon-\varphi(y)}$$
$$\times\frac{\partial}{\partial\varepsilon}\left[\pm\sqrt{\varepsilon-\varphi(y)}\left(f_0^{\pm} + \frac{df_0^{\pm}}{d\varepsilon}\right) + cf_3^{\pm}\right], \tag{32}$$

where $f_0^{\pm}(\epsilon)$ is the distribution function for the outer region in zeroth order in ν, as given by Eq. (24). Making some elementary transformations, we get

$$\dot{W}_{\pm}^{(1)} = -\frac{\omega T v_T c}{\sqrt{8\pi}}\nu\varphi_0^{1/2}\left\{\left(4-\frac{\pi^2}{2}\right)f_0^{\pm}(\varphi_0)\right.$$
$$\left.+2\int_1^{\infty} dt\cdot t^{-1/2}f_0(\varphi_0 t)\left[K(t^{-1/2}) - \frac{\pi^2}{4}E^{-1}(t^{-1/2})\right]\right\}, \tag{33}$$

where K(k) and E(k) are the complete elliptic integrals of first and second kinds with the modulus $k = t^{-1/2}$.

In the integral in the right member of Eq. (33) we can go to the limit $\varphi_0 \to 0$ under the integral sign, since the resulting integral converges. We get the result

$$\dot{W}^{(1)} = \dot{W}_{+}^{(1)} + \dot{W}_{-}^{(2)} = -\frac{\omega Tnce^{-c^2}}{4\pi^{3/2}}\nu\varphi_0^{1/2}$$
$$\times\left\{4-\frac{\pi^2}{2} + 2\int_1^{\infty}\frac{dt}{t^{1/2}}\left[K(t^{-1/2}) - \frac{\pi^2}{4}E^{-1}(t^{-1/2})\right]\right\}. \tag{34}$$

Using the properties of elliptic integrals, we can reduce the integral in the right member of Eq. (34) to the form

$$J = \pi - 2 - \frac{\pi}{2}\int_0^1\frac{dt}{t^{3/2}E(t)}\left[E(t) - \frac{\pi}{2}\right].$$

The last integral can be calculated approximately, for example by expanding the integrand in series. To 1 per cent accuracy it is equal to ½. Putting this in Eq. (34), we get the final expression for the contribution of the outer region to the damping of the wave:

$$\dot{W}^{(1)} = -\tfrac{1}{16}(6-\pi)\,\pi^{-1/2}\nu\varphi_0^{1/2}ce^{-c^2}\omega Tn. \tag{35}$$

2. Let us now examine the contribution from the inner region $\epsilon < \varphi_0$. Using Eq. (29) and noting that for $\epsilon < \varphi_0$ we have

$$f^+(\varepsilon, y) = f^-(\varepsilon, y),$$

we can write

$$\dot{W}^{(2)} = \frac{\omega T v_T}{\sqrt{2\pi}}\int_{-\pi}^{\pi} dy\int_{\varphi(y)}^{\varphi_0} d\varepsilon\,[\varepsilon-\varphi(y)]^{-1/2}\varphi'(y)\,f(y, \varepsilon). \tag{36}$$

Substituting the expression (26) for $f(y, \epsilon)$ in Eq. (36) and confining ourselves to terms of lowest order in ν and φ_0 (in this approximation $e^{-\epsilon} \approx 1$), we get after elementary transformations

$$\dot{W}^{(2)} = -2\pi^{-3/2}\nu\varphi_0^{3/2}\omega Tnc. \tag{37}$$

Since the contribution to the damping from the outer region is $\sim\varphi_0^{1/2}$, we can neglect the contribution from the inner region, which is of order $\varphi_0^{3/2}$.

3. Let us now consider the contribution from the boundary layer (Fig. 2), where we cannot use the expansion of the distribution function in powers of ν. Although the width of this region is very small ($\sim\nu$), as will be seen below its contribution to the damping is of the same order of magnitude as that from the outer region (since the second derivative of the distribution function with respect to ϵ takes large values in the boundary layer).

We shall show that for the calculation of the contributions from the boundary layer it is not necessary to know the form of the distribution function in this layer; it is enough to have the limiting values, in zeroth approximation in ν, of the distribution function and its first derivative at the boundaries between the boundary layer and the outer and inner regions. On the basis of Eq. (29) we can write

$$\dot{W}^{(3)} = \frac{\omega T v_T}{\sqrt{8}\pi} \int_{-\pi}^{\pi} dy \int_{\varphi_0-\delta}^{\varphi_0+\delta} \frac{d\varepsilon}{\sqrt{\varepsilon - \varphi(y)}} \varphi'(y) (f^+ + f^-) = -\frac{\omega T v_T}{\sqrt{8}\pi}$$
$$\times \int_{-\pi}^{\pi} dy \int_{\varphi_0-\delta}^{\varphi_0+\delta} d\varepsilon \frac{d(\varepsilon - \varphi(y))^{1/2}}{dy} [f^+(y, \varepsilon) + f^-(y, \varepsilon)], \quad (38)$$

where $2\delta(y)$ is the width of the boundary layer.

Integrating by parts with respect to y, using the periodicity of the integrand in this variable, dropping terms of order $\varphi_0^{3/2}$, and inserting $\partial f^\pm/\partial y$ from Eq. (18), we get

$$\dot{W}_\pm^{(3)} = \frac{\omega T v_T}{\sqrt{2}\pi} \nu \int_{-\pi}^{\pi} dy \left\{ \int_{\varphi_0-\delta}^{\varphi_0+\delta} d\varepsilon \sqrt{\varepsilon - \varphi(y)} \frac{\partial}{\partial \varepsilon} [(c \pm \sqrt{\varepsilon - \varphi}) f^\pm \right.$$
$$\left. \pm \int_{\varphi_0-\delta}^{\varphi_0+\delta} d\varepsilon \sqrt{\varepsilon - \varphi} \frac{\partial}{\partial \varepsilon} \left(\sqrt{\varepsilon - \varphi} \frac{\partial f^\pm}{\partial \varepsilon} \right) \right\},$$
$$\dot{W}^{(3)} = \dot{W}_+^{(3)} + \dot{W}_-^{(3)}. \quad (39)$$

In the boundary layer the first derivative $\partial f/\partial \epsilon$ is of higher order in ν than the second derivative. Therefore we can neglect the first integral in Eq. (39) in comparison with the second. Using the fact that to our degree of accuracy the limiting values of f at the boundaries between the boundary layer and the outer and inner regions are equal, we can easily transform the second integral in Eq. (39) into the form

$$\int_{\varphi_0-\delta}^{\varphi_0+\delta} d\varepsilon (\varepsilon - \varphi) \frac{\partial^2 f^\pm}{\partial \varepsilon^2} = \int_{\varphi_0-\delta}^{\varphi_0+\delta} d\varepsilon \frac{\partial}{\partial \varepsilon} \left[(\varepsilon - \varphi) \frac{\partial f^\pm}{\partial \varepsilon} \right]$$
$$= \varphi_0 \cos^2 \frac{y}{2} \left(\left.\frac{\partial f_0^\pm}{\partial \varepsilon}\right|_{\varepsilon = \varphi_0 + 0} - \left.\frac{\partial f_0^\pm}{\partial \varepsilon}\right|_{\varepsilon = \varphi_0 - 0} \right). \quad (40)$$

Inserting here the limiting values of the derivatives $\partial f_0^\pm/\partial \epsilon$ at the edges of the outer and inner regions from Eqs. (24a) and (26), and then substituting Eq. (40) in Eq. (39), we get

$$\dot{W}^{(3)} = -\tfrac{1}{2} \sqrt{\pi}\, \nu \varphi_0^{1/2} e^{-c^2} c \omega T n. \quad (41)$$

Combining the quantities (35) and (41), we get the expression for the damping decrement $\gamma = -\dot{W}/2\overline{W}$ (where $\overline{W} = E_0^2/16\pi = (k\Phi_0)^2/64\pi$ is the mean energy density of the wave):

$$\gamma = \frac{12\alpha}{\tau_D} \left(\frac{v_f}{v_T} \right)^4 \exp\left(-\frac{v_f^2}{2v_T^2} \right) \left(\frac{e\Phi_0}{T} \right)^{-3/2}; \quad \alpha = \frac{7\pi + 6}{16\sqrt{\pi}}. \quad (42)$$

For simplicity we have set $\omega = \omega_0$ and used Eq. (19). We see that the damping of the wave vanishes for $\tau_D \to \infty$, i.e., when the collision frequency goes to zero.

Unlike the Landau decrement (2), in our case γ depends on the amplitude of the wave through the factor $\Phi_0^{-3/2}$. Expressing γ in terms of the Landau decrement (2), we get

$$\gamma = \beta \left(\frac{v_T}{v_f} \right)^2 \frac{\tau_2}{\tau_1} \gamma_L, \quad \beta = \frac{3\sqrt{2}(7\pi + 6)}{4\pi^2} \approx 3.0, \quad (43)$$

where we have introduced two characteristic times: τ_1, the time for establishing local equilibrium in the resonance region Δv, Eq. (1), owing to Coulomb collisions (with scattering through small angles),

$$\tau_1 = \tau_D (\Delta v)^2/v_f^2 = \tau_D (v_T/v_f)^2 e\Phi_0/T \quad (44)$$

[τ_D is given by Eq. (10)], and τ_2, the time of the nonlinear distortion of the distribution function under the action of the field of the wave [2,8]

$$\tau_2 = \lambda (e\Phi_0/m)^{-1/2} = (2\pi/kv_T)(e\Phi_0/T)^{-1/2} \quad (45)$$

(λ is the wavelength).

In our limiting case of sufficiently large waves (cf. Eq. (16)) we must take[1)]

$$\tau_2 \ll \tau_D e\Phi_0/T = \tau_1 (v_f/v_T)^2, \quad (46)$$

i.e., the time of the nonlinear distortion must be less than the time for establishing the equilibrium distribution in the resonance region through collisions. Furthermore the damping decrement (43) is much smaller than the Landau decrement. We must also call attention to the fact that it follows from Eqs. (43) and (44) that the local equilibrium in the resonance region is established much more quickly than the wave is damped.

We point out finally that under the condition (46) the formula (43) agrees with the qualitative estimate of the damping decrement for plane waves obtained in [2] (see page 96) on the basis of the results of the quasi-linear theory.

Let us now examine the question of the permissibility of neglecting the time derivative in the kinetic equation (11). Assuming that in Eq. (11)

$$\partial f/\partial x \sim fk/2\pi, \quad \partial f/\partial v \sim f (e\Phi/m)^{-1/2}$$

[1)]It must be pointed out that the inequalities (16) and (46) are not completely equivalent: (46) is stronger. As an accurate estimate of the two members of Eq. (18) shows, the more rigorous condition for the applicability of the expansion in powers of ν is (46), and not (16). The physical meaning of the conditions (16) and (46) is that the amplitude of the wave must greatly exceed the thermal noise level. In the opposite limiting case the damping is that given by Landau.

and using Eq. (46), we find that the time derivative is indeed small in comparison with the other terms in the kinetic equation if $\gamma \ll \tau_1^{-1}$, that is, the time for damping of the wave must be much larger than the time for establishing equilibrium in the resonance region owing to collisions. A comparison of Eqs. (43) and (44) shows that this condition is satisfied.

In conclusion the authors express their gratitude to R. Z. Sagdeev for valuable advice and fruitful discussions of the questions treated in this paper, to A. A. Vedenov and E. P. Velikhov for helpful discussions of the results, and also to I. O. Foreskin for some comments.

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Translated by W. H. Furry
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