IV. MHD STABILITY OF A LONGITUDINAL PINCH.

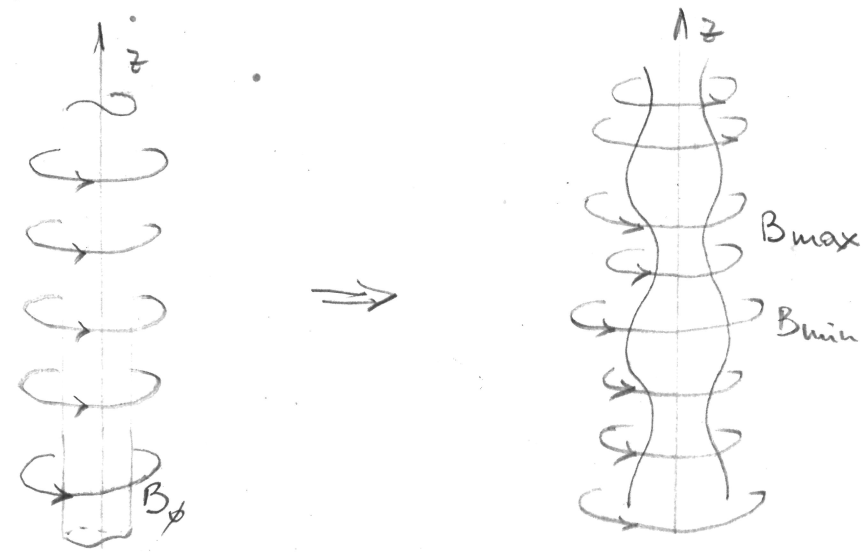
In order to represent a toroidal equilibrium of circular cross-section with $R_0/a \gg 1$ ($R_0 \equiv$ mayor radius of the torus, $a \equiv$ minor radius) or a very elongated cylindrical equilibrium, it is quite common to use an infinite cylindrical model in which all equilibrium quantities depend only on the radius (distance from the symmetry axis. We have already shown that in this case the equilibrium is described by the following equation:

$$\nabla p - \frac{(\nabla\times\underline{B})\times\underline{B}}{4\pi} \equiv \hat{e}_r\left[\frac{d}{dr}\left(p + \frac{B_z^2+B_\phi^2}{8\pi}\right) + \frac{B_\phi^2}{4\pi r}\right] = 0 \qquad (53)$$

where $\underline{B} = B_z(r)\hat{e}_z + B_\phi(r)\hat{e}_\phi$. In order to study the stability of these equilibria we will use the Energy Principle by Bernstein et al. [Proc. of the Royal Soc., A 244 (1958) 17.]. However, it can be useful, first to get some physical insight.

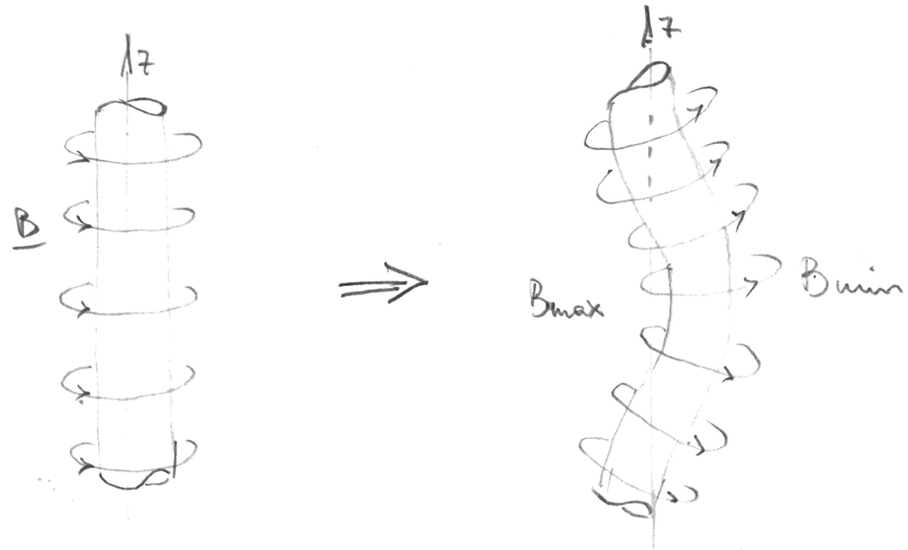
When $\underline{B_\phi = 0}$ (ideal θ-pinch) the pinch is absolutely stable (within the limits of ideal MHD), since the $\underline{B}$ lines are straight, only stable oscillations are possible.

When $\underline{B_z = 0}$ (ideal z-pinch) it is easy to see that this kind of equilibria is unstable. For example, let us consider a deformation of the plasma boundary like $\underline{\xi} \propto \hat{e}_r\left(e^{i(m\phi+kz)} + c.c.\right)$ in the case $m=0$ we have:



B_ϕ^2 will be greater where the pinch is constricted, this in turn will generate a pressure that tend to increase the deformation, giving rise to the so-called "<u>sausage instability</u>" that will destroy the plasma column.

In the case $m = 1$ $k \neq 0$ we have:



Also in these case the difference of magnetic pressure will give rise to an unstable motion known as "<u>kink instability</u>". These instabilities can be suppressed by adding a B_z field inside the plasma, the resulting pinch (mixing between θ and z pinches) is named stabilized pinch. Anyway, its stability analysis is not so simple and in order to get some conclusions it is helpful to analyze a cylindrical pinch with $B_z \neq 0$ and $B_\phi \neq 0$, in which the plasma is in contact with a perfectly conducting and rigid wall of radius R. This last assumption greatly simplifies the use of the Energy Principle (no exterior problem has to be solved). It is quite usual in MHD theory first to analyze perturbations that do not deform the plasma boundary (this is equivalent to assume the plasma in contact with a perfectly rigid and conducting wall), if instability can be shown, surely the pinch will be unstable also in the absence of these restrictions.

If the plasma boundary is not deformed, the Energy Principle gives the following expression for the potential energy associated to an infinitesimal displacement $\xi(r,\phi,z)$ of the plasma fluid elements:

$$\delta W = \frac{1}{2} \int_{V_{plasma}} d^3x \left\{ \gamma p (\nabla\cdot\underline{\xi})^2 + \nabla\cdot\underline{\xi}\; \underline{\xi}\cdot\nabla p + \frac{[\nabla\times(\underline{\xi}\times\underline{B})]^2}{4\pi} - \frac{\underline{\xi}\times\nabla\times\underline{B}\cdot\nabla\times\underline{\xi}\times\underline{B}}{4\pi} \right\} \tag{54}$$

where p and $\underline{B}$ are equilibrium quantities, $\gamma = c_p/c_v$ and the integration is extended to all the plasma volume. $\underline{\xi}$ has to be physically possible and tangent to the plasma boundary. $\delta W < 0$ for some $\underline{\xi}$ will implies instability. The techinque consists in looking for extrema of δW with respect to arbitrary but finite variations of $\underline{\xi}$. If the extrema of δW can be negative, instability will be demonstrated, but if they are positive no conclusion can be obtained since δW may have no minima. The method is the Euler-Lagrange method of extremization of functionals and it gives only necessary, and not sufficient, conditions for the existence of extrema.

Since the equilibrium depends only on r we can use $\underline{\xi}$ of the kind:

$$\underline{\xi} = \frac{1}{2} \sum_{m,k} \left[\underline{\xi}_{m,k}(r)\, e^{i(m\phi + kz)} + c.c. \right] \qquad \begin{pmatrix} m \text{ integer} \\ -\infty < k < \infty \end{pmatrix} . \tag{55}$$

Owing to the ortonormality of $e^{i(m\phi + kz)}$ δW can be written as:

$$\delta W = \sum_{m,k} \delta W_{m,k} \, ,$$

Therefore, δW will be negative if and only if at least one $\delta W_{m,k} < 0$. In the following we will omit the indexes m and k and analyze only one Fourier-component of ξ, moreover, let us indicate $\frac{d}{dr}\xi = {}'$ and consider δW as the energy change per unit length of the pinch:

$$\begin{aligned} \delta W = \frac{1}{8} \int_0^R dr\, r \Big\{ & \alpha^2 \underline{\xi}\cdot\underline{\xi}^* + \left(B_z^2 + B_\phi^2 + 4\pi\gamma p\right) |\nabla\cdot\underline{\xi}|^2 \\ & - 2\frac{B_\phi^2}{r} \left(\nabla\cdot\underline{\xi}\, \xi_r^* + \nabla\cdot\underline{\xi}^*\, \xi_r\right) - 2 B_\phi \left(\frac{B_\phi}{r}\right)' \xi_r \xi_r^* + i\alpha \Big[\frac{2 B_\phi}{r} \left(\xi_r^* \xi_\phi \right. \\ & \left. - \xi_\phi^* \xi_r\right) - B_\phi \left(\xi_\phi \nabla\cdot\underline{\xi}^* - \xi_\phi^* \nabla\cdot\underline{\xi}\right) - B_z \left(\xi_z \nabla\cdot\underline{\xi}^* - \xi_z^* \nabla\cdot\underline{\xi}\right) \Big] \Big\} . \end{aligned} \tag{56}$$

Where we have used (53) and:

$$\alpha = \frac{m B_\phi}{r} + k B_z \quad .$$

As it can be seen, $\delta W = \int_0^r dr\, \mathcal{G}(\xi_\phi, \xi_\phi^*, \xi_z, \xi_z^*; \xi_r, \xi_r^*, \xi_r', \xi_r^{*\prime}, r)$ is a functional of ξ, ξ_r' and c.c. but not of ξ_ϕ' and ξ_z', in such a way that the ξ_ϕ and ξ_z that extremize δW have to satisfy the following equations:

$$\frac{\partial \mathcal{G}}{\partial \xi_\phi} = \frac{\partial \mathcal{G}}{\partial \xi_\phi^*} = 0 \equiv \frac{r}{8}\Big\{ \alpha^2 \xi_\phi^* + (B_\phi^2 + B_z^2 + 4\pi\gamma p)\frac{im}{r}\nabla\cdot\xi^* - \frac{2im}{r^2} B_\phi^2 \xi_r^*$$
$$+ i\alpha\Big[2\frac{B_\phi}{r}\xi_r^* - B_\phi \nabla\cdot\xi^* + \frac{im}{r} B_\phi \xi_\phi^* + \frac{im}{r} B_z \xi_z^* \Big]\Big\} \quad , \qquad (57)$$

$$\frac{\partial \mathcal{G}}{\partial \xi_z} = \frac{\partial \mathcal{G}}{\partial \xi_z^*} = 0 \equiv \frac{r}{8}\Big\{ \alpha^2 \xi_z^* + ik(B_\phi^2 + B_z^2 + 4\pi\gamma p)\nabla\cdot\xi^* - \frac{2ik}{r} B_\phi^2 \xi_r^*$$
$$+ i\alpha\Big[-B_z \nabla\cdot\xi^* + ik B_\phi \xi_\phi^* + ik B_z \xi_z^* \Big]\Big\} \quad , \qquad (58)$$

where we used the fact that $\nabla\cdot\xi = \frac{(r\xi_r)'}{r} + \frac{im}{r}\xi_\phi + ik\xi_z$.

By adding $(57)\cdot B_\phi$ with $(58)\cdot B_z$ it results:

$$4\pi\gamma p \left(\frac{m B_\phi}{r} + k B_z\right)\nabla\cdot\xi^* = 0 \quad , \qquad (59)$$

from which it results that δW should be minimum when $\nabla\cdot\xi = 0$. That we are in the presence of a minimum can be seen by computing $\frac{\partial^2 \mathcal{G}}{\partial\xi_\phi \partial\xi_\phi^*}$ and $\frac{\partial^2 \mathcal{G}}{\partial\xi_z \partial\xi_z^*}$ and observing that when (59) is satisfied they are positive. In general (59) can be satisfied also when $p = 0$ (trivial case), or when $\alpha = \frac{m B_\phi}{r} + k B_z = 0$, this can be satisfied always in some points of the plasma since m and k are arbitrary, but only for very special distribution of B_ϕ and B_z it can be satisfied in all the plasma. Therefore, it is reasonable to assume that the most dangerous ξ should be divergence free in all the plasma (incompressible displacement).

Employing $\nabla\cdot\xi = \nabla\cdot\xi^* = 0$ eq. (56) can be reduced to:

$$\delta W = \frac{1}{8}\int_0^R dr\, r \left\{ \alpha^2 \xi\cdot\xi^* - 2B_\phi\left(\frac{B_\phi}{r}\right)' \xi_r \xi_r^* + i\alpha\left[\xi_r^*\xi_\phi - \xi_r\xi_\phi^*\right]\frac{2B_\phi}{r}\right\}. \qquad (60)$$

We can eliminate the dependence on ξ_ϕ and ξ_z by using (57) and (58). Adding (57) $\cdot\, \frac{8\xi_\phi}{r}$ with (58) $\cdot\, \frac{8\xi_z}{r}$ we obtain

$$\alpha^2\left(\xi_\phi\xi_\phi^* + \xi_z\xi_z^*\right) = -2\frac{B_\phi^2}{r}\,\xi_r^*(r\xi_r)' - i\alpha\Big[\frac{2B_\phi}{r}\xi_r^*\xi_\phi - \frac{(r\xi_r)'}{r}\left(B_\phi\xi_\phi^* + B_z\xi_z^*\right)\Big].$$

From $\nabla\cdot\xi = 0 \rightarrow \xi_\phi^* = -\frac{i}{m}(r\xi_r^*)' - \frac{kr}{m}\xi_z^*$, introducing it into (58) we obtain ξ_z^* as a function of ξ_r^* and $\xi_r^{*\prime}$:

$$\xi_z^* = \frac{ik}{\alpha\left(\frac{m}{r} + \frac{k^2 r}{m}\right)}\left[\left(\frac{B_\phi}{r} - \frac{kB_z}{m}\right)\xi_r^* - \frac{\alpha r}{m}\xi_r^{*\prime}\right]$$

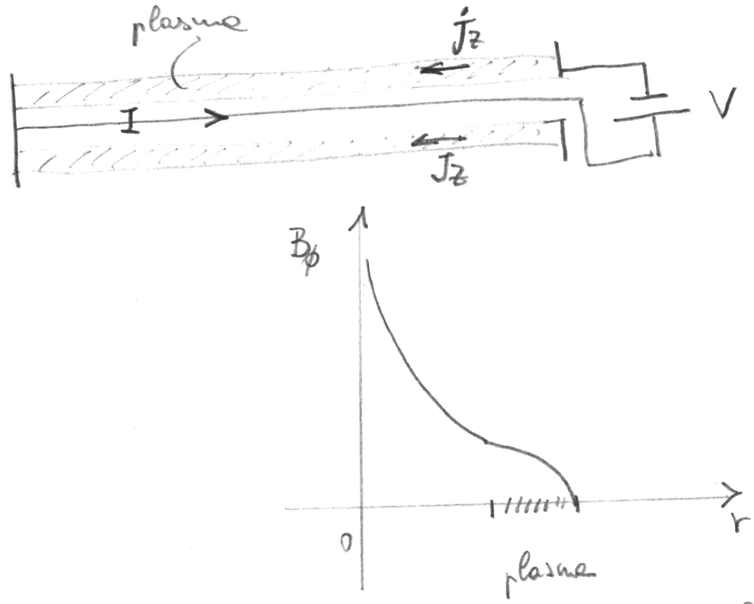
and introducing it into the expression for ξ_ϕ^* it results:

$$\xi_\phi^* = -\frac{i}{m}(r\xi_r^*)' - \frac{ik^2 r}{m\alpha\left(\frac{m}{r} + \frac{k^2 r}{m}\right)}\left[\left(\frac{B_\phi}{r} - \frac{kB_z}{m}\right)\xi_r^* - \frac{\alpha r}{m}\xi_r^{*\prime}\right].$$

After a straightforward algebra it is possible to obtain δW as a functional of $\xi_r, \xi_r', \xi_r^*, \xi_r^{*\prime}$ only:

$$\delta W = \frac{1}{8}\int_0^R dr\, r\left\{\xi^2\left[\alpha^2 - \frac{2B_\phi}{r^2}(rB_\phi)'\right] + \frac{1}{(m^2 + k^2r^2)}\left(\beta\xi + \alpha r\xi'\right)^2\right\}, \qquad (61)$$

where we restricted to real ξ_r, suppressed the sub-index r and introduced $\beta = \left(\frac{mB_\phi}{r} - kB_z\right)$ for simplicity. From (61) it is evident that an ideal θ-pinch will be stable, since the only term that can be negative is associated to B_ϕ $\left[-\xi^2\frac{2B_\phi}{r}(rB_\phi)'\right]$. In general, in a Z-pinch $\frac{(rB_\phi)'}{B_\phi} > 0$, and δW can be negative. $\frac{(rB_\phi)'}{B_\phi} < 0$ can be satisfied in a hollow Z-pinch where all the plasma longitudinal current returns along a conductor located inside the plasma column. Such devices exist and are known as "hard-core" pinches :



Such devices need also a B_z in order to confine the plasma, in them the plasma can be quite stable but their usefulness are limited by the problem of impurities.

From (61) we can obtain some interesting results. Let us first consider perturbations with;

$\underline{m \neq 0}$: Let us consider perturbations with different values of m and k, but with the same $q = k/m$. The only term in (61) containing m is $\alpha^2 = m^2 \left(\frac{B_\phi}{r} + q B_z \right)^2$, being positive, δW will be minimum when $m^2 = 1$. Therefore, for a given $q = k/m$, the most dangerous perturbation will correspond to $m = \pm 1$ (kink instability), if a pinch is stable for $m^2 = 1$ and $-\infty < k < \infty$ it will be stable for all $m^2 > 1$.

$\underline{m = 0}$: In this case δW reduces to:

$$\delta W = \frac{1}{8} \int_0^R dr\, r \left\{ k^2 \xi^2 B_z^2 - \frac{2 B_\phi (r B_\phi)'}{r^2} \xi^2 + \frac{(r B_z \xi' - B_z \xi)^2}{r^2} \right\}$$

From it, it results that an ideal z-pinch is unstable to $m = 0$ perturbations (sausage inst.), but it can be stabilized adding a B_z (all terms including B_z are positive). Moreover, if we have stability for $k^2 \to 0$, the pinch will remain stable for all k. (this is true for $\nabla\cdot\xi = 0$, but it can be shown that it is also valid when $m = k = 0$ and $\nabla\cdot\xi \neq 0$, since $\nabla\cdot\xi \neq 0$ add positive terms to δW)

In the following we will restrict the analysis to $m \neq 0$ perturbations, since they are the most common instabilities observed in stabilized z-pinches and in toroidal devices. The fact is that $m=0$ instabilities can be suppressed by a sufficiently strong B_z, but this does not prevent $m \neq 0$ unstable motions.

We can write δW as

$$\delta W = \int_{r_1}^{r_2} dr \, \mathcal{Y}(\xi, \xi', r) \qquad . \qquad (62)$$

The ξ that makes δW extremum must satisfy the following variational principle. Let us introduce a small variation $\delta\xi$ of ξ such that $\delta\xi(r_1) = \delta\xi(r_2) = 0$, then the corresponding variation of δW, $\Delta(\delta W)$, can be expressed to the 1st order in $\delta\xi$ as:

$$\Delta(\delta W) = \int_{r_1}^{r_2} dr \left(\delta\xi \frac{\partial \mathcal{Y}}{\partial \xi} + \delta\xi' \frac{\partial \mathcal{Y}}{\partial \xi'} \right) =$$

$$= \int_{r_1}^{r_2} dr \, \delta\xi \left(\frac{\partial \mathcal{Y}}{\partial \xi} - \frac{\partial^2 \mathcal{Y}}{\partial r \partial \xi'} \right) + \underbrace{\cancel{\delta\xi \frac{\partial \mathcal{Y}}{\partial \xi'} \Big|_{r_1}^{r_2}}}_{=0} \quad .$$

In order δW be an extremum $\longrightarrow$ $\Delta(\delta W) = 0$. Since $\delta\xi$ is arbitrary it follows:

$$\frac{\partial \mathcal{Y}}{\partial \xi} - \frac{\partial^2 \mathcal{Y}}{\partial r \partial \xi'} = 0 \qquad . \qquad (63)$$

(63) is the Euler-Lagrange equation to which the ξ, making δW extremum, must obey. Such extreme can be maxima, minima but also inflection points, therefore (63) has to be intended as

a necessary condition for the existence of a minimum of δW.

Using (61), eq (63) is of the kind:

$$\xi'' + P\xi' + Q\xi = 0 \tag{64}$$

where

$$P = \frac{3}{r} + \frac{2\alpha'}{\alpha} - \frac{2k^2 r}{m^2 + k^2 r^2}$$

$$Q = -\left(k^2 + \frac{m^2-1}{r^2}\right) - \frac{2k^2\beta}{\alpha(m^2+k^2r^2)} - \frac{8\pi k^2 p'}{\alpha^2 r}$$

Since δW can be written as:

$$\delta W = \int_0^R dr \left(A\xi^2 + 2H\xi\xi' + B\xi'^2\right)$$

if ξ is a solution of (63) it implies $A\xi + H\xi' = (H\xi + B\xi')'$ in such a way that

$$\delta W = \int_0^R dr \left[\xi(A\xi + H\xi') + \xi'(H\xi + B\xi')\right] =$$

$$= \int_0^R dr \left[\xi(H\xi + B\xi')' + \xi'(H\xi + B\xi')\right] =$$

$$= \xi(H\xi + B\xi')\Big|_0^R \equiv 0$$

since $\xi(R) = 0$ and near $r=0$, $H \propto r^2$ and $B \propto r^3$, δW results identically vanishing. However, such integration is uncorrect, since being m and k arbitrary, there can exist some point inside the plasma where P and Q in eq. (64) may be singular, i.e. points where $\alpha = 0$. This implies that the solutions of (64) cannot be continued across the singular points and δW must be expressed as a sum of integrals with the singular points as limits of integration.

$\alpha = 0$ implies $m\frac{B_\phi}{r} = -kB_z \rightarrow \frac{B_\phi}{B_z} = -\frac{kr}{m}$, if we consider the $\underline{B}$ line equation:

$$\frac{B_\theta}{B_z} = \frac{r\,d\phi}{dz} = -\frac{kr}{m}$$

it follows:

$$m\,d\phi + k\,dz = 0 \quad ,$$

which means that the amplitude of ξ is constant along the $\underline{B}$ lines lying on the magnetic surface satisfying $\alpha = 0$.
In order to express δW when there exists some point $r = a$ for which $\alpha(a) = 0$, it is necessary to study the behaviour of the solution of eq (64) near such point.

When $\alpha = 0$ eq (64) has a singular point of the Fuchsian kind, since P hase a simple pole and Q has a double pole.
From the theory of linear differential equations we know that near $r = a$ ξ should be like:

$$\xi \sim (r-a)^{\nu}$$

with ν solution of the indicial equation:

$$\nu^2 + (p_0 - 1)\nu + q_0 = 0 \quad , \tag{65}$$

where

$$p_0 = \lim_{r \to a} (r-a)\, P(r)$$

$$q_0 = \lim_{r \to a} (r-a)^2\, Q(r)$$

Since, near $r = a$ $P(r) \sim \frac{2\alpha'}{\alpha}$ and writing $\alpha \sim \alpha'(r-a)$ it follows:

$$p_0 = 2 \quad ,$$

and analogously

$$q_0 = -\frac{8\pi k^2 p'}{a\,\alpha'^2(a)} = M \quad ,$$

in general $M > 0$ since $p' < 0$ in general. Therefore, the indicial equation will be:

$$\nu^2 + \nu + M = 0 \tag{66}$$

from which $\nu = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4M}$.

Two cases can be distinguished:

i) $4M > 1$: the two roots are complex, they have negative real part and ξ will be like:

$$\xi \propto |r-a|^{-1/2} \cos\left(\frac{\sqrt{4M-1}}{2} \ln|r-a|\right)$$

for $r \to a$ ξ diverges and oscillates indefinitely.

ii) $4M \leq 1$, the two roots are real and at least one is negative. In this case for $r \to a$ ξ diverges but it does not oscillate.

Both cases are not physically realizable. However, B. Suydam (Proc. of the 2nd United Nations International Conf. on Peaceful Uses of Atomic Energy, Geneva 1958, p. 157) introduced, in the case of only one singular point (but the analysis can be extended to several singular points), the following displacement:

$$\xi = \xi_{E1} \quad \text{for} \quad 0 \leq r \leq a-\varepsilon$$

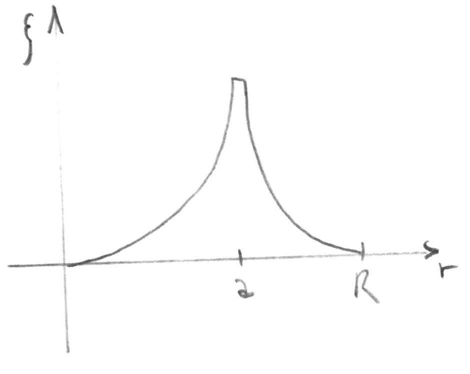
$$\xi = \xi_0 \quad '' \quad a-\varepsilon < r < a+\varepsilon$$

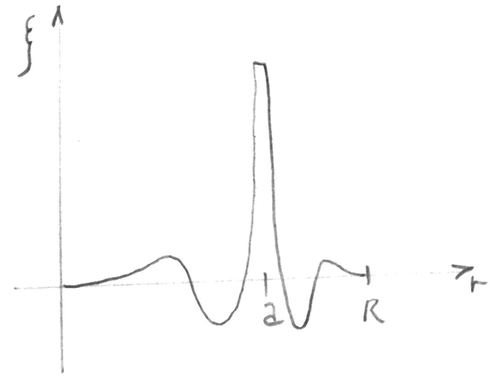
$$\xi = \xi_{E2} \quad '' \quad a-\varepsilon \leq r \leq R$$

with ε infinitesimal, ξ_{E1} solution of the Euler-Lagrange equation (69) in the inner zone, ξ_0 constant and ξ_{E2} solution of the Euler-Lagrange equation in the outer zone. Moreover $\xi_{E2}(R) = 0$ and

$$\xi_{E1}(a-\varepsilon) = \xi_0 = \xi_{E2}(a+\varepsilon)$$

Examples of such displacement are:





The corresponding δW is:

$$\delta W = \delta W_1 + \delta W_2 + \delta W_3 \quad ,$$

where

$$\delta W_1 = \int_0^{a-\varepsilon} dr\, \mathcal{F}\left(\xi_{E1}, \xi'_{E1}, r\right) = \xi_{E1}\left(H\xi_{E1} + B\xi'_{E1}\right)\Big|_{r=a-\varepsilon}$$

$$\delta W_2 = 2\varepsilon\, \mathcal{F}\left(\xi_0, 0, a\right) = 2\varepsilon\, A(a)\, \xi_0^2$$

$$\delta W_3 = \int_{a+\varepsilon}^{R} dr\, \mathcal{F}\left(\xi_{E2}, \xi'_{E2}, r\right) = -\xi_{E2}\left(H\xi_{E2} + B\xi'_{E1}\right)\Big|_{r=a+\varepsilon}$$

In case ii) we obtain:

$$4M \leq 1 \implies \quad \xi_{E1,2} \propto (r-a)^{\nu} \implies \xi'_{E1,2} \propto \nu (r-a)^{\nu-1}$$

$$\xi'_{E1}(a-\varepsilon) = -\frac{\nu \xi_0}{\varepsilon} \quad ; \quad \xi'_{E2}(a+\varepsilon) = \frac{\nu \xi_0}{\varepsilon}$$

and:

$$A(a) = a\left[-\frac{2B_\phi(a)\left(rB_\phi\right)'\big|_{r=a}}{a^2} + \frac{\beta^2(a)}{m^2+k^2a^2}\right]$$

$$H(a \mp \varepsilon) = \mp \frac{\varepsilon\, a^2 \alpha'(a) \beta(a)}{m^2+k^2a^2} + \mathcal{O}(\varepsilon^2)$$

$$B(a \mp \varepsilon) = \frac{\varepsilon^2 \alpha'^2(a)\, a^3}{m^2+k^2a^2} + \mathcal{O}(\varepsilon^3)$$

$$\alpha'(a) = m\, B_z(a)\left(\frac{B_\phi}{rB_z}\right)'\Bigg|_{r=a}$$

and δW reduces to:

$$\delta W = 2\varepsilon a^3 \xi_0^2 B_z^2\left[\left(B_\phi/rB_z\right)'\big|_{r=a}\right]^2\left(|\nu| - M\right) \quad ,$$

since $|\nu| = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4M} \longrightarrow |\nu| - M = \frac{1}{2} - M \pm \frac{\sqrt{1-4M}}{2} =$
$= \frac{1}{4}\left(1 \pm \sqrt{1-4M}\right)^2 \geq 0$, it follows $\delta W \geq 0$, i.e. stability.

In case i) we have:

$4M > 1 \Rightarrow$ ν complex of the kind $\nu_{1,2} = -\frac{1}{2} \pm i\frac{\gamma}{2}$ with $\gamma = \sqrt{4M-1}$. For $r \to a$ we have:

$$\xi_{E\,1,2} \propto |r-a|^{-1/2} \cos\left(\frac{1}{2}\gamma \ln|r-a| + \phi_0\right) ,$$

where ϕ_0 has to be chosen in such a way that:

$$\xi_{E\,1,2}(a \mp \varepsilon) = \xi_0 .$$

Introducing $\lambda = \frac{1}{2}\gamma \ln|r-a| + \phi_0$ we can express:

$$\xi'_{E\,1,2}(a \mp \varepsilon) \propto \mp \frac{\cos\lambda}{2\varepsilon^{3/2}} - \frac{\sin\lambda\,\lambda'}{\sqrt{\varepsilon}} = \mp \frac{1}{2\varepsilon^{3/2}}\left(\cos\lambda \pm \gamma \sin\lambda\right)$$

and

$$\xi^2_{E\,1,2}(a \pm \varepsilon) = \xi_0^2 \propto \frac{\cos^2\lambda}{\varepsilon} .$$

Taking into account the results of case ii), δW results:

$$\delta W \propto \frac{a^3 \alpha'^2(a)}{(m^2+k^2a^2)}\left[\sqrt{4M-1}\,\sin 2\lambda + (1-2M)(1+\cos 2\lambda)\right] .$$

When $\varepsilon \to 0$, $\lambda \to \infty$ and the quantity

$$\sqrt{4M-1}\,\sin 2\lambda + (1-2M)(1+\cos 2\lambda) \qquad (67)$$

can change sign. In effect it has maxima and minima in the points satisfying:

$$\operatorname{tg} 2\lambda = \frac{\sqrt{4M-1}}{1-2M} \quad \begin{cases} \sin 2\lambda = \pm \dfrac{\sqrt{4M-1}}{2M} \\ \cos 2\lambda = \pm \dfrac{(1-2M)}{2M} \end{cases}$$

and (67) will oscillate between 1 and $(1-4M) < 0$, therefore δW can be negative and the pinch will be unstable to $m \neq 0$ perturbations.

Remembering that $M = -\dfrac{8\pi k^2 p'}{a\,\alpha'^2(a)}$ and introducing the quantity $\mu = \dfrac{B_\phi}{rB_z} = \dfrac{d\phi}{dz}$ being $\dfrac{dz}{d\phi}$ the pitch of a magnetic line on a cylindrical magnetic surface. We can express $\alpha'(a)$

as:

$$\alpha'\Big|_{r=a} = \left(\frac{m B_\phi}{r} + k B_z\right)'\Big|_{r=a} = \left[B_z\left(\frac{m B_\phi}{r B_z} + k\right)\right]'\Big|_{r=a} =$$

$$= \left(B_z\, m\, \mu'\right)\Big|_{r=a}$$

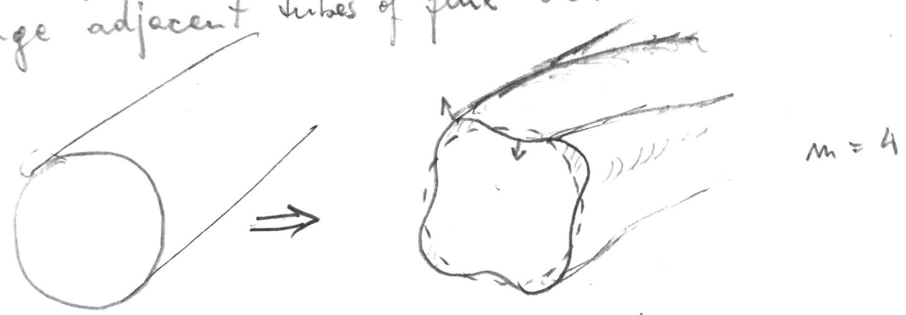
moreover in $r=a$ $k^2 = m^2\mu^2$ and $4M \leq 1$ (careli), results equivalent to:

$$\boxed{\frac{8\pi p'}{B_z^2} + \frac{r}{4}\left(\frac{\mu'}{\mu}\right)^2 \geqslant 0} \quad , \qquad (68)$$

eq. (68) has to be intended has a necessary condition for stability to $m \neq 0$ perturbations, and it is widely known as

"<u>Suydam Criterion</u>".

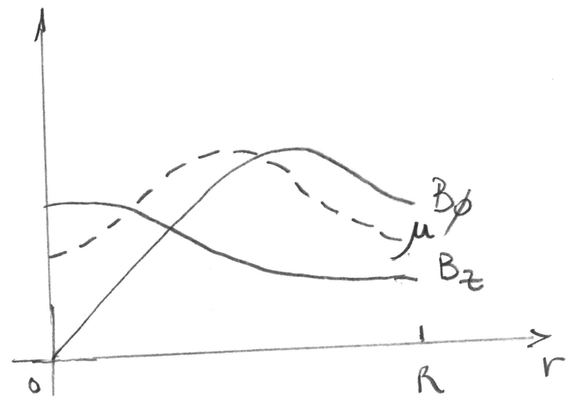
It is a local condition and allows to get several insights into the better way to confine a plasma. As it has been shown the most dangerous perturbations are resonant with magnetic lines on a certain magnetic surface where the amplitude of ξ grows in a divergent way. Such kind of modes ~~are~~ known as interchange or flute modes, they interchange adjacent tubes of flux across the resonant surface:



looking at (68) since in general $p' < 0$ it can be seen that the stabilizing term μ'^2 depends on the change of pitch of the magnetic lines, much greater is this change more stable should be the pinch. This change of pitch with radius is known as <u>magnetic shear</u>, and the stability can be interpreted in terms of the increased magnetic distortion that must be introduced when magnetic tubes are interchanged in the presence of shear. Heuristically we can think that the motion of the plasma is induced by p' but it is obstaculated by the magnetic shear. In general a mode should develop essentially near a resonant surface in its early phase and then propagate throughout the plasma reaching a macroscopic deformation of the column. The most dangerous mode should correspond to $m = \pm 1$ as it is experimentally observed (kink modes).

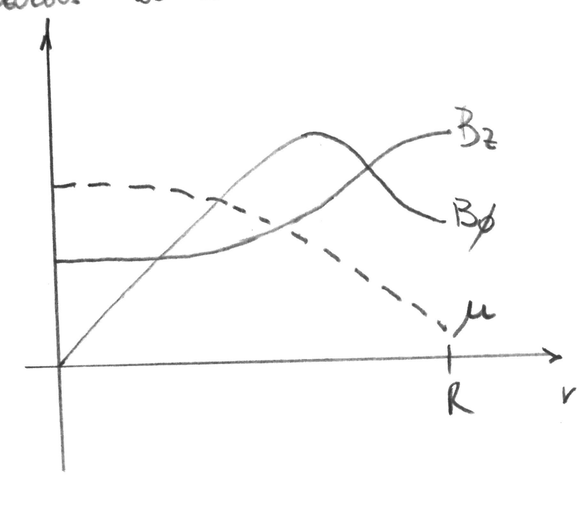
The Suydam criterion is useful in the evaluation of some concepts for the magnetic confinement of plasmas. Let us suppose to have a stabilized Z-pinch in which B_z does not change sign. Near $r=0$ B_z is approximately constant and it is also constant near the wall, $B_\phi \propto r$ near $r=0$ and $B_\phi \propto 1/r$ near the wall. If B_z decreases in going from $r=0$ to the wall we have, qualitatively, the following situation:

$$\mu = \frac{B_\phi}{r B_z}$$



In general μ has a maximum inside the plasma and there, even a small p' will give rise to instability.

If B_z increases towards the wall we will have something like

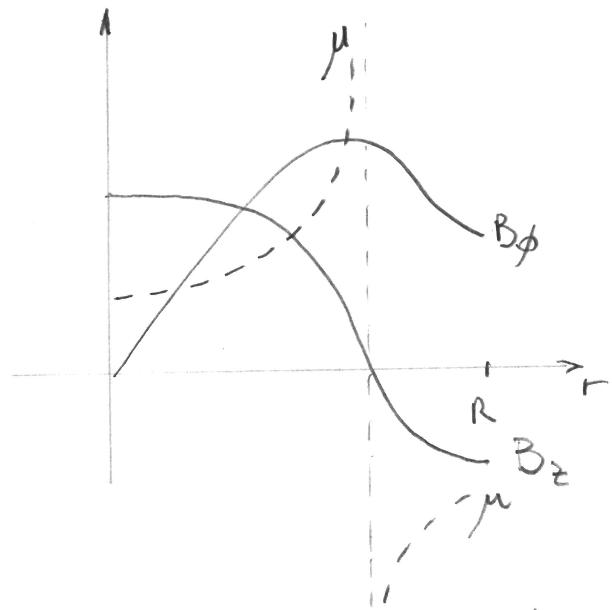


and $\mu' \neq 0$ in all the plasma is in principle possible. This second case should be a better candidate, from the point of view of stability, for magnetic confinement than the former configuration. The concept is experimentally obtained in the <u>screw-pinches</u> both longitudinal or toroidal ones (z-discharge accompanied by a Θ-pinch compression).

There is another interesting situation when B_z change sign in going from $r=0$ to the wall. In this case we have:

$\mu' \sim \frac{B_\phi B_z'}{r B_z^2}$

$\frac{\mu'}{\mu} \sim \frac{B_z'}{B_z}$



$\mu' \neq 0$ in all the plasma and it can be much greater than in the screw-pinch concept. (the stability near $B_z = 0$ strongly depends on p' but it can be satisfied by obtaining the inversion sufficiently near the wall where p' is small). This magnetic field configuration is successfully used in the <u>reversed-field pinches</u> (both cylindrical or toroidal) by properly shaping the currents in the external coils. They have quite good characteristics of stability (better than screw-pinches) and are serious candidates for future fusion reactor based in magnetic confinement, since their β is quite high, $\sim 20\%$.

From all the analysis here presented it is also possible to visualize how the tokamak concept has arose. The concept relies ~~on~~ forbidding $m=1$ instability in the column. To show how to do this we have to remember that a circular cross-section torus of very large aspect ratio ($R_0/a \gg 1$ with R_0 major radius and a minor radius can be approximated by a cylinder of finite length $2\pi R_0$, and we have to identify the two cylinder ends. This means that ξ must be the same at $z=0$ and $z=2\pi R_0$, which implies $k = \ell/R_0$ with ℓ integer. In order to develop the most dangerous $|m|=1$ instability, there must exist some radius for which

$$\frac{B_\phi}{r} + \frac{\ell}{R_0} B_z = 0 \quad \rightarrow \quad \ell = -\frac{R_0}{r}\frac{B_\phi}{B_z} .$$

If we are able to create

$$\left|\frac{B_z}{B_\phi}\right| > \frac{R_0}{r}$$

in all the plasma, the equality $\ell = -\frac{R_0}{r}\frac{B_\phi}{B_z}$ can not be satisfied since ℓ must be zero or integer. This in turn means that we cannot have magnetic surfaces which are resonant with $|m|=1$ perturbations and therefore no kink instability. However $|m|>1$ modes may still be unstable but in general they are less dangerous in experiments.

Let us return to the magnetic line equation:

$$\frac{r\,d\phi}{dz} = \frac{B_\phi}{B_z} ,$$

if we increment dz in $2\pi R_0$ the corresponding increment in $d\phi$ will be:

$$\Delta\phi = \frac{2\pi R_0}{r}\frac{B_\phi}{B_z} = \frac{2\pi}{q}$$

with $|q|>1$ when $\left|\frac{B_z}{B_\phi}\right| > \frac{R_0}{r}$. $|q|>1$ implies that magnetic lines do not close after a circulation around the simmetry axis of the torus, if $q=n$ integer, they close after n circulations, but in general q will be not an integer and this means that a magnetic line will cover ergodically a magnetic surface (this happens also when $q = \frac{n_1}{n_2} < 1$ with $n_2 > n_1$ integers). The q concept can be extended to two-dimensional (r,z)

equilibria and $\iota = \frac{2\pi}{q}$ is known as rotational transform, it depends only on Ψ, the magnetic surface. In tokamaks in general $q(r=0) \gtrsim 1$ and $q(r=a) \sim 3$. In the literature $q = 1$ is known as Kruskal-Shafranov limit, in order to have a chance to be MHD stable, tokamaks have to operate above it. (toroidal pinches typically operate below $q = 1$, and stability is due essentially to the shear). Since in a tokamak the pressure gradient is balanced essentially by B_ϕ (in the cylindrical approximation) we have $p_{max} \sim \frac{B_\phi^2(a)}{8\pi}$, moreover since $B_z \sim q(a) B_\phi(a) \frac{R_0}{a}$ is practically constant inside the plasma, we have

$$\beta \sim \frac{8\pi\, p_{max}}{\left(B_z^2 + B_\phi^2\right)} \sim \frac{1}{q^2 \frac{R_0^2}{a^2} + 1} << 1 \quad (\text{typically } 1\%)$$

and that is why tokamaks are low β configurations. High β ($\sim 10\%$) configurations are possible only with $q < 1$ and their stability relies on the magnetic shear, since perturbations resonant with magnetic surfaces are always possible.

All the analysis for the cylindrical pinch can be extended to 2-dimensional equilibria, thanks to a formidable amount of algebra, the equivalent criterion is the Mercier criterion, but it depends on several integration on a given magnetic surface, in general its evaluation has to be done numerically. However, the concepts of magnetic shear and Kruskal-Shafranov limit are not altered in their essence.