Implied Stochastic Volatility Models

Yacine Aït-Sahalia   Chenxu Li   Chen Xu Li

Princeton University and NBER   Peking University   Princeton University

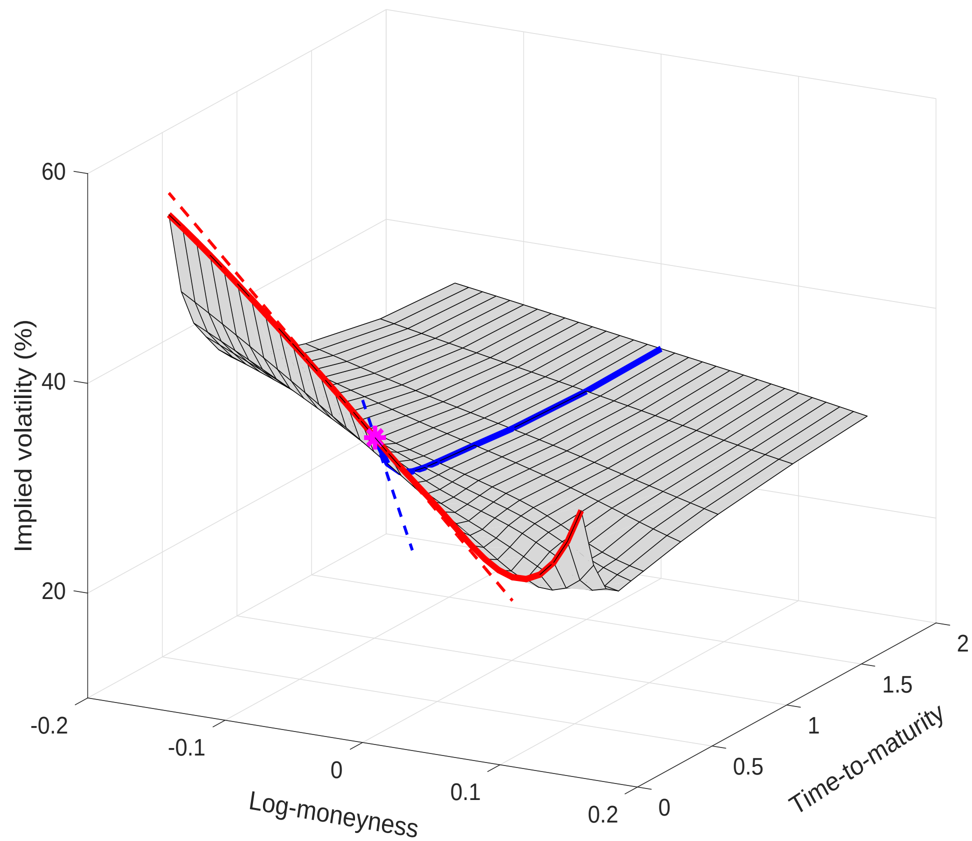
1. Motivation

- Data = option prices or equivalently implied volatility surface (IV)
- Models that makes sense of the data = stochastic volatility (SV)
- Link between the two?
  - Not explicit
  - Relies on numerical computation of option prices
  - By numerical Fourier inversion in the easiest cases (affine)
  - Or binomial tree approximations, PDE numerical methods, or Monte Carlo simulations
  - Followed by numerical computation of the IV
  - Yet IV is how prices are quoted, while SV is how options are priced, hedged, etc.

- Example model: SV model of Heston (1993), parameters $\theta = (\kappa, \alpha, \xi, \rho)$

$$dS_t/S_t = (r-d)dt + \sqrt{V_t}dW_{1t}$$
$$dV_t = \kappa(\alpha - V_t)dt + \xi\sqrt{V_t}(\rho dW_{1t} + \sqrt{1-\rho^2}dW_{2t})$$

- Example data: S&P500 index options IV surface on May 20, 2010



- Standard estimation method:

$$\min_{\theta} \sum_{i=1}^{n} \left(P_i^{\text{data}} - P^{\text{model}}\left(S_i, K_i, \tau_i, \theta\right) \right)^2$$

- Requires the numerical computation of P^{model} each time the minimization algorithm adjusts θ

- Minimization on top of the re-pricing at the slightly altered θ values very difficult, especially outside of the affine class of models

- This paper:
  - A new method that uses directly the information contained in IV data to construct a SV model, or estimate an existing one
  - Relies on a small number of observable shape characteristics of the IV surface
  - No need to compute prices numerically – all is in closed form
  - We call the resulting models implied stochastic volatility models, since their coefficient functions have been constructed to reproduce the desired features of the IV surface
  - In the Black-Scholes setting, implied volatility is a single parameter; in the SV setting, the quantity analogous to implied volatility needs to be an entire SV model.

- IV shape characteristics
  - Level, slope and curvature along log-moneyness and term structure dimensions
  - Useful descriptions of the IV data when trading/pricing options: straddle, risk-reversal and butterfly-spread are functions of at-the-money level, slope, and convexity in log-moneyness; calendar-spread depends on the term structure slope.

$$\frac{dS_t}{S_t} = (r - d)dt + v_t dW_{1t},$$
$$dv_t = \mu(v_t)dt + \gamma(v_t)dW_{1t} + \eta(v_t)dW_{2t}.$$

- Simple to implement: Closed form, even for non-affine models commonly regarded as analytically intractable
- First, regress the IV data on log-moneyness and time-to-expiration to estimate the IV shape characteristics
- Nonparametrically: The coefficient functions $\mu(\cdot)$, $\gamma(\cdot)$ and $\eta(\cdot)$ of the SV model can be recovered from the IV shape characteristics by local polynomial regression
- Parametrically: Use the observed shape characteristics as a set of GMM conditions to estimate the parameters
- Also of interest: leverage effect coefficient function $\rho(v_t) = \frac{\gamma(v_t)}{\sqrt{\gamma(v_t)^2+\eta(v_t)^2}}$

- ISVM empirical results
  - Strong state-dependent leverage effect, mean reversion in volatility, monotonicity and state dependency in volatility of volatility.
  - Matches well other characteristics of the IV surface not employed as inputs.
  - Stable over time.
  - Performs well out of sample.

2. Literature

- Many empirical studies: Bates (2000), Chernov et al. (2003), Aït-Sahalia and Kimmel (2007), Christoffersen et al. (2009), Egloff et al. (2010), etc.

- Option price approximations: Jarrow and Rudd (1982), Yoshida (1992), Fouque et al. (2000), Kunitomo and Takahashi (2001), Kunitomo and Takahashi (2003), Benhamou et al. (2010), Kristensen and Mele (2011), Li (2014), Xiu (2014)

- Implied volatility approximations: Hagan and Woodward (1999), Lewis (2000), Berestycki et al. (2004), Medvedev and Scaillet (2007), Henry-Labordère (2008), Forde et al. (2012), Gatheral et al. (2012), Takahashi and Yamada (2012), Gao and Lee (2014), Jacquier and Lorig (2015).

3. Linking SV and IV

3.1. From SV to IV

$$\frac{dS_t}{S_t} = (r-d)dt + v_t dW_{1t},$$
$$dv_t = \mu(v_t)dt + \gamma(v_t)dW_{1t} + \eta(v_t)dW_{2t}.$$

- $P(\tau, k, S_t, v_t)$: price of a European put with time-to-maturity $\tau = T - t$ and exercise strike K, log-moneyness $k = \log(K/S_t)$

$$P(\tau, k, S_t, v_t) = e^{-r\tau}\mathbb{E}_t[\max(S_t e^k - S_T, 0)],$$

- $\Sigma(\tau, k, v_t)$: implied volatility, solves

$$P_{\mathrm{BS}}(\tau, k, S_t, \Sigma) = P(\tau, k, S_t, v_t).$$

- $\Sigma_{i,j}$: shape characteristics of the IV surface (local level, monotonicity, convexity)

$$\Sigma_{i,j}(v_t) = \lim_{\tau \to 0} \frac{\partial^{i+j}}{\partial \tau^i \partial k^j} \Sigma(\tau, 0, v_t).$$

- $\Sigma_{0,0}$ = at-the-money level of the IV surface

$$\Sigma_{0,0}(v_t) = v_t,$$

- This is a known limit: Ledoit et al. (2002) and Durrleman (2008)

- Next, we need the slope $\Sigma_{0,1}$, and convexity $\Sigma_{0,2}$ along the log-moneyness dimension, and the slope $\Sigma_{1,0}$ along the term-structure dimension

- We show that

$$\Sigma_{0,1}(v_t) = \frac{1}{2v_t}\gamma(v_t)$$
$$\Sigma_{0,2}(v_t) = \frac{1}{6v_t^3}[2v_t\gamma(v_t)\gamma'(v_t) + 2\eta(v_t)^2 - 3\gamma(v_t)^2],$$
$$\Sigma_{1,0}(v_t) = \frac{1}{24v_t}[2\gamma(v_t)(6(d-r) - 2v_t\gamma'(v_t) + 3v_t^2) + 12v_t\mu(v_t) + 3\gamma(v_t)^2 + 2\eta(v_t)^2].$$

- These formulae apply to all existing one-factor SV models, such as:
  - The Heston model, $v(x) = \sqrt{x}$, $\mu(x) = \kappa(\alpha - x)$, $\gamma(x) = \xi\rho\sqrt{x}$, and $\eta(x) = \xi\sqrt{1-\rho^2}\sqrt{x}$,
  - The GARCH-SV model, $v(x) = \sqrt{x}$, $\mu(x) = \kappa(\alpha - x)$, $\gamma(x) = \xi\rho x$, and $\eta(x) = \xi\sqrt{1-\rho^2}x$,
  - The log-linear SV model, $v(x) = \exp(x)$, $\mu(x) = \kappa(\alpha - x)$, $\gamma(x) = (\xi_0 + \xi_1 x)\rho$, and $\eta(x) = (\xi_0 + \xi_1 x)\sqrt{1-\rho^2}$.
  - etc.

3.2. From IV to SV

- The idea: invert these formulae back into the unknown coefficients functions of the SV model, $\mu(\cdot)$, $\gamma(\cdot)$, and $\eta(\cdot)$
- No further approximation, numerical solution of a differential equation or other numerical inversion is required.
- $v_t = \Sigma_{0,0}(v_t)$ and we show that

$$\gamma(v_t) = 2\Sigma_{0,0}(v_t)\Sigma_{0,1}(v_t),$$

and

$$\eta(v_t) = \Big[6\Sigma_{0,0}(v_t)^3\Sigma_{0,2}(v_t) + 8\Sigma_{0,0}(v_t)^2\Sigma_{0,1}(v_t)^2 - 16\Sigma_{0,0}(v_t)^2\Sigma_{0,1}(v_t)\Sigma'_{0,1}(v_t)\Big]^{-1/2},$$

$$\mu(v_t) = \Sigma_{0,0}(v_t)^2\left[\Sigma_{0,1}(v_t)(2\Sigma'_{0,1}(v_t) - 1) - \frac{1}{4}\Sigma_{0,2}(v_t)\right] - 2(d-r)\Sigma_{0,1}(v_t) + 2\Sigma_{1,0}(v_t).$$

- This new result characterizes the closed-form relationship between the SV model coefficient functions and the IV shape characteristics:

$$\mu(\cdot), \gamma(\cdot), \eta(\cdot) \quad \leftrightarrows \quad \Sigma_{0,1}(\cdot), \Sigma_{0,2}(\cdot), \Sigma_{1,0}(\cdot)$$

- Some implications

  - Steeper log-moneynessslope $\Sigma_{0,1}(v_t)$ results in higher vol-of-vol $\gamma(v_t)$

  - The sign of the leverage effect coefficient $\rho(v_t)$ is determined by the sign of the slope $\Sigma_{0,1}(v_t)$: downward-sloping IV smile, $\Sigma_{0,1}(v_t) < 0$, translates into $\rho(v_t) < 0$.

  - Greater convexity $\Sigma_{0,2}(v_t)$ results in larger vol-of-vol and weaker leverage effect $\rho(v_t)$

  - Higher term-structure slope $\Sigma_{1,0}(v_t)$results in an increase in the drift $\mu(v_t)$, i.e., a faster expected change of the instantaneous volatility v_t.

4. From data to model: Nonparametric ISVM

- The objective is to identify the unknown coefficient functions μ, γ, and η
- Start by estimating the shape characteristics $\Sigma_{i,j}$ from the observed IV surfaces
  - Polynomial regression of IV on time-to-maturity τ and log-moneyness k

$$\Sigma^{\text{data}}(\tau_l, k_l) = \sum_{j=0}^{J} \sum_{i=0}^{L_j} \beta_l^{(i,j)} (\tau_l)^i (k_l)^j + \epsilon_l$$

  - We obtain

$$[\Sigma_{i,j}]_l^{\text{data}} = i!j!\hat{\beta}_l^{(i,j)}$$

and in particular, $v_{l\Delta} = [\Sigma_{0,0}]_l^{\text{data}} = \hat{\beta}_l^{(0,0)}$.

- While the objects of interest $\Sigma_{i,j}$ are derivatives of the IV surface Σ evaluated at $(\tau, k) = (0, 0)$, the regression includes observations with (τ, k) away from $(0, 0)$ in order to estimate these partial derivatives.

- We then use the system of equations given above

$$[\gamma]_l^{\text{data}} = 2[\Sigma_{0,0}]_l^{\text{data}}[\Sigma_{0,1}]_l^{\text{data}},$$

and regress nonparametrically

$$[\gamma]_l^{\text{data}} = \gamma(v_{l\Delta}) + \epsilon_l,$$

- To estimate the coefficient functions $\eta(\cdot)$ and $\mu(\cdot)$, we need γ and its derivative γ'.
- We use locally linear kernel regression (see, e.g., Fan and Gijbels (1996))
- Locally linear kernel regression provides in one pass an estimator of the regression function and of its derivative:

$$[\gamma]_l^{\text{data}} \approx \alpha_0 + \alpha_1(v_{l\Delta} - v) + \epsilon_l,$$

$$\hat{\gamma}(v) = \hat{\alpha}_0 \text{ and } \hat{\gamma}'(v) = \hat{\alpha}_1.$$

- Then we plug-in:

$$
\begin{aligned}
[\eta]_l^{\text{data}} = \Big[2 \Big(& 3([\Sigma_{0,0}]_l^{\text{data}})^3 [\Sigma_{0,2}]_l^{\text{data}} \\
& - 2\, [\Sigma_{0,0}]_l^{\text{data}} \hat{\gamma}(v_{l\Delta}) \hat{\gamma}'(v_{l\Delta}) + 3\hat{\gamma}(v_{l\Delta})^2 \Big) \Big]^{-1/2},
\end{aligned}
$$

- And

$$
\begin{aligned}
[\mu]_l^{\text{data}} &= 2[\Sigma_{1,0}]_l^{\text{data}} + \frac{\hat{\gamma}(v_{l\Delta})}{6} (2\hat{\gamma}'(v_{l\Delta}) - 3[\Sigma_{0,0}]_l^{\text{data}}) \\
&\quad - \frac{\hat{\eta}(v_{l\Delta})^2}{6[\Sigma_{0,0}]_l^{\text{data}}} - \frac{\hat{\gamma}(v_{l\Delta})}{[\Sigma_{0,0}]_l^{\text{data}}} \left(d - r + \frac{1}{4} \hat{\gamma}(v_{l\Delta}) \right).
\end{aligned}
$$

- To summarize: estimate the shape characteristics from the IV surface data via standard regression; then, recover the SV coefficient functions nonparametrically by locally linear regression and plug into the closed-form relations.

5. Parametric ISVM

- $\mu(\cdot) = \mu(\cdot;\theta)$, $\gamma(\cdot) = \gamma(\cdot;\theta)$, and $\eta(\cdot) = \eta(\cdot;\theta)$

- Form moment conditions:

$$g^{(i,j)}(v_{l\Delta};\theta) = [\Sigma_{i,j}]_l^{\text{data}} - [\Sigma_{i,j}(v_{l\Delta};\theta)]^{\text{model}}$$

- $[\Sigma_{i,j}]_l^{\text{data}}$ = obtained by the same regression of IV on k and τ as above, and closed-form formulae for $[\Sigma_{i,j}(v_{l\Delta};\theta)]^{\text{model}}$ given above

- Followed by standard GMM

- Monte Carlo results in the paper

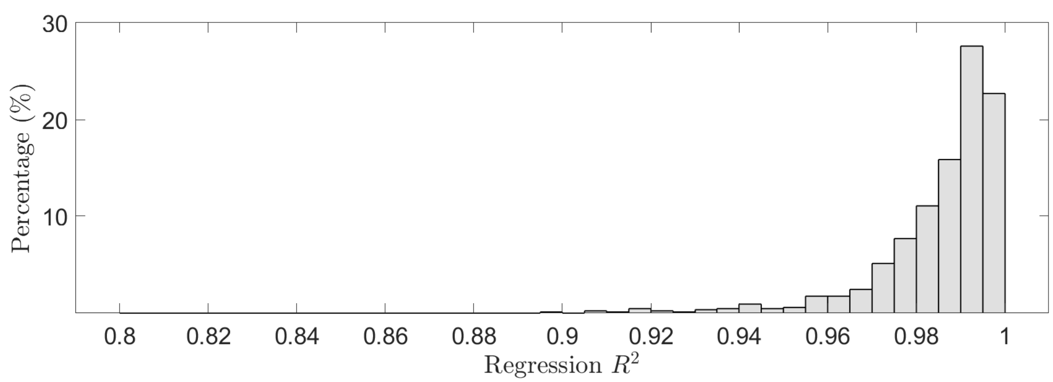
- Bootstrap standard errors

6. Empirical results

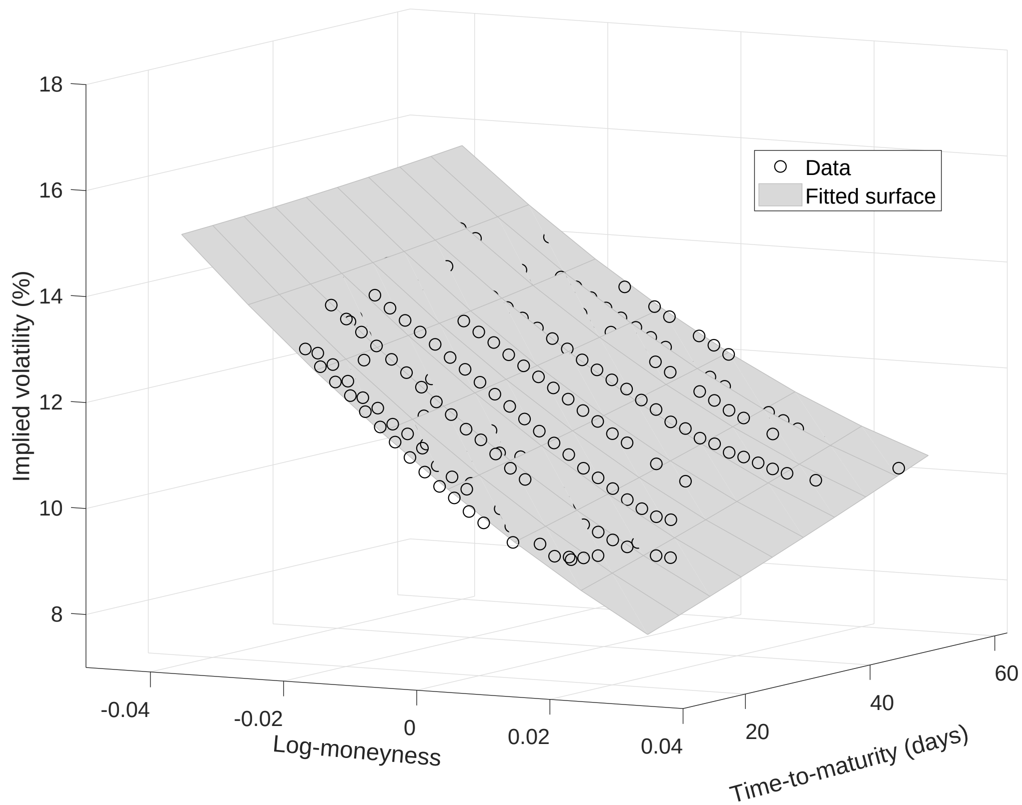
- S&P 500 options, January 2, 2013 - December 29, 2017, daily frequency, time-to-maturity between 15 and 60 calendar days, 269,622 observations

- IV surface regression

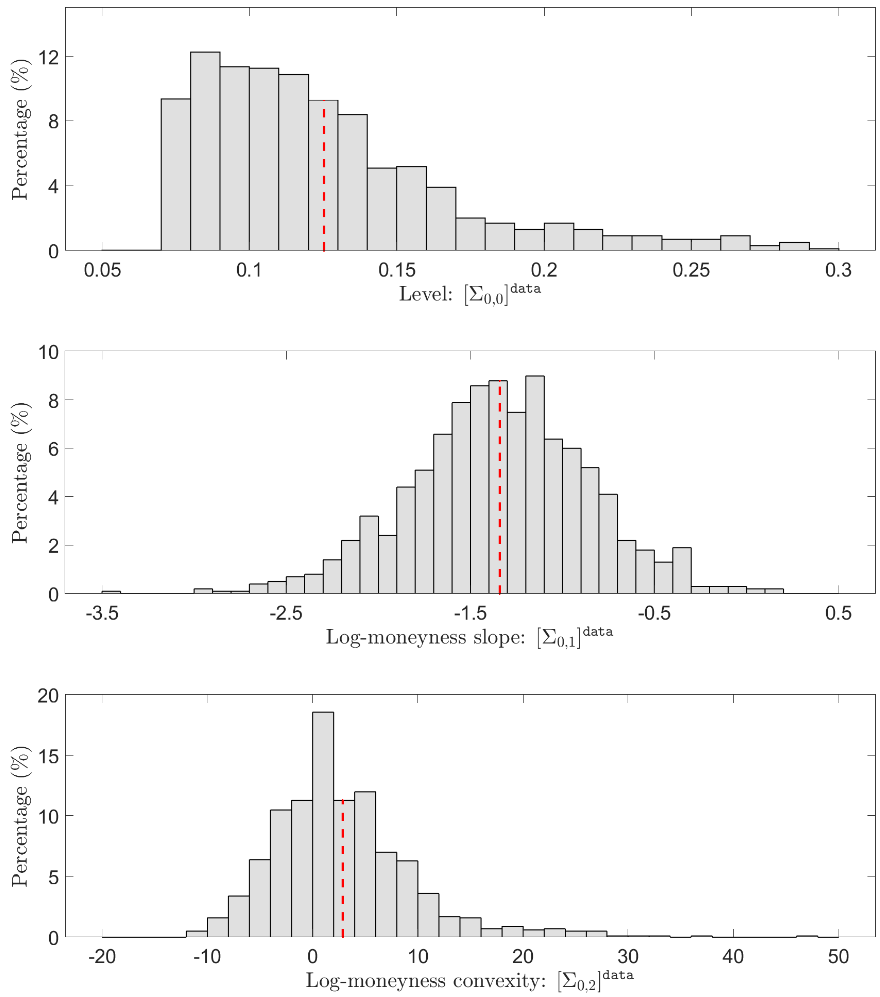
$$\Sigma^{\text{data}}(\tau_l, k_l) = \beta_l^{(0,0)} + \beta_l^{(1,0)}\tau_l + \beta_l^{(2,0)}(\tau_l)^2 + \beta_l^{(0,1)}k_l + \beta_l^{(1,1)}\tau_l k_l + \beta_l^{(2,1)}(\tau_l)^2 k_l + \beta_l^{(0,2)}(k_l)^2 + \epsilon_l.$$



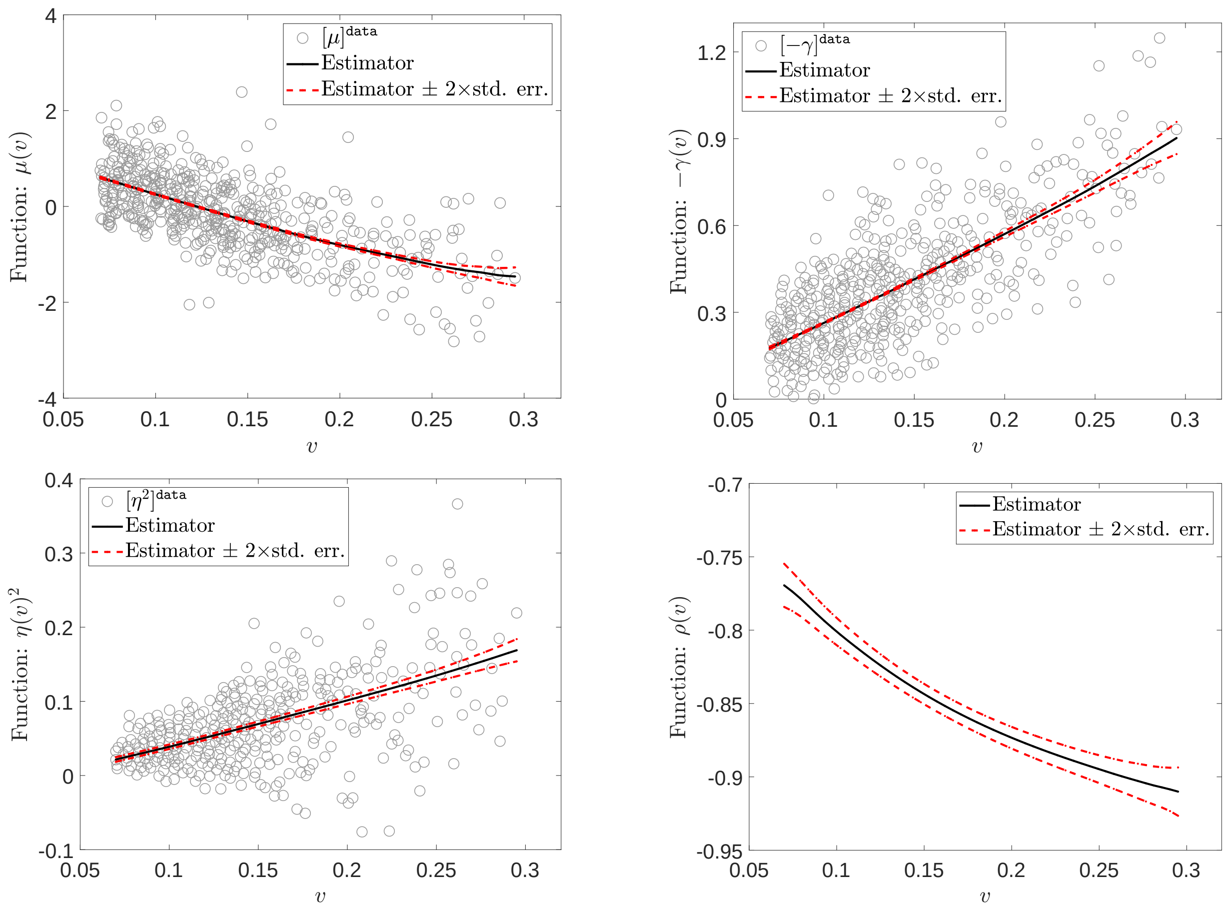
Example: January 3, 2017



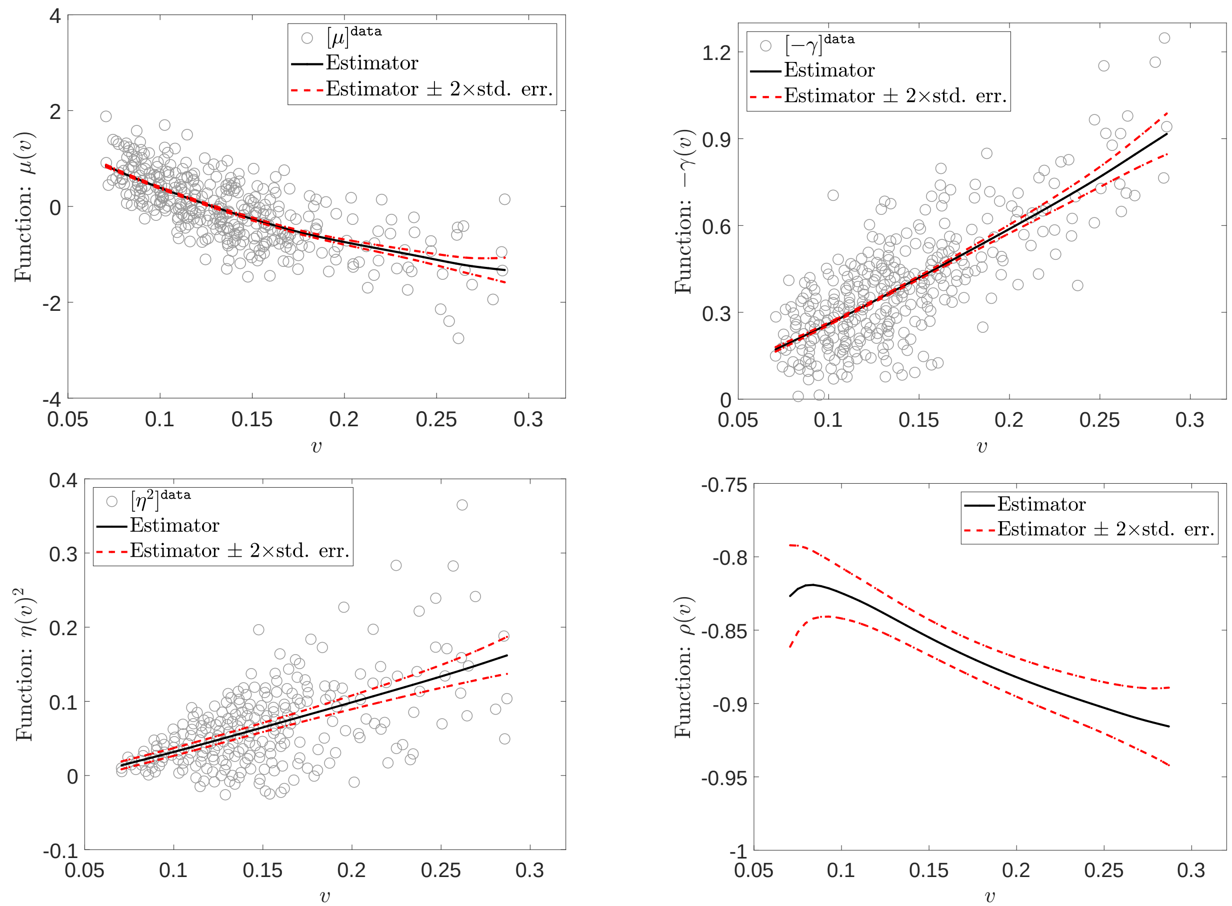
Time series distribution of IV shape characteristics



Full sample 2013-17



In-sample 2013-15



7. Adding jumps

$$\frac{dS_t}{S_{t-}} = (r - d - \lambda(v_t)\bar{\mu})dt + v_t dW_{1t} + (\exp(J_t) - 1)dN_t,$$
$$dv_t = \mu(v_t)dt + \gamma(v_t)dW_{1t} + \eta(v_t)dW_{2t}.$$

- N_t is a doubly stochastic Poisson process (Cox process) with stochastic intensity $\lambda(v_t)$
- J_t is the size of log-price jump, assumed independent of the asset price$S$
- Objective: recover $\mu(\cdot)$, $\gamma(\cdot)$, $\eta(\cdot)$, $\lambda(\cdot)$ and the law of J

- The bivariate expansion of the IV surface can be generalized to allow for jumps
  - incorporating the square root of time-to-maturity $\sqrt{\tau}$
  - as well as negative powers of $\sqrt{\tau}$

$$\Sigma^{(J,\mathbf{L}(J))}(\tau, k, v_t) \;=\; \sum_{j=0}^{J} \sum_{i=\min(0,1-j)}^{L_j} \varphi^{(i,j)}(v_t)\tau^{\frac{i}{2}}k^j,$$

- Closed-form expressions for the coefficients of this expansion are given in the paper

- $\gamma(v_t)$, $\eta(v_t)$ and $\mu(v_t)$ are all affected by the presence of jumps
  - The third order partial derivative $\partial^3\Sigma/\partial k^2\partial\tau$ (and the term-structure slope $\partial\Sigma/\partial\tau$ and $\partial^2\Sigma/\partial\tau^2$) enter
  - In the continuous case, the term-structure slope $\partial\Sigma/\partial\tau$ was the only IV characteristic along the term-structure dimension that mattered

- The paper shows that it is theoretically possible to recover $\mu(\cdot)$, $\gamma(\cdot)$, $\eta(\cdot)$, $\lambda(\cdot)$ and the law of J from the IV shape characteristics using the same ideas as in the continuous case
- In practice
  - Estimating third order shape characteristics of the IV surface accurately is not possible given the limitations of the data currently available: A substantially denser set of observations would be necessary
  - The divergence of the IV surface due to the presence of negative powers of τ also requires very short maturity options to be accurately observed

8. Conclusions

- New method: The SV coefficient functions of an ISVM are estimated by exploiting the restrictions provided by the IV shape characteristics, nonparametrically or parametrically

- Whether the model is analytically tractable or not (i.e., affine or not) does not constrain the method.

- GMM conditions are fully explicit and require no other computations (such as option prices) that are inherently difficult in other methods

- Nonparametric ISVM estimated in the data with a strong state-dependent leverage effect, mean reversion in volatility, monotonicity and state dependency in volatility of volatility; stable over time and performing well out of sample.

References

Aït-Sahalia, Y., Kimmel, R., 2007. Maximum likelihood estimation of stochastic volatility models. Journal of Financial Economics 83, 413–452.

Bates, D. S., 2000. Post-'87 crash fears in the S&P 500 futures option market. Journal of Econometrics 94, 181–238.

Benhamou, E., Gobet, E., Miri, M., 2010. Time dependent Heston model. SIAM Journal on Financial Mathematics 1, 289–325.

Berestycki, H., Busca, J., Florent, I., 2004. Computing the implied volatility in stochastic volatility models. Communications on Pure and Applied Mathematics 57, 1352–1373.

Chernov, M., Gallant, A. R., Ghysels, E., Tauchen, G. T., 2003. Alternative models for stock price dynamics. Journal of Econometrics 116, 225–257.

Christoffersen, P., Heston, S., Jacobs, K., 2009. The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well. Management Science 55, 1914–1932.

Durrleman, V., 2008. Convergence of at-the-money implied volatilities to the spot volatility. Journal of Applied Probability 45, 542–550.

Egloff, D., Leippold, M., Wu, L., 2010. The term structure of variance swap rates and optimal variance swap investments. Journal of Financial and Quantitative Analysis 45, 1279–1310.

Fan, J., Gijbels, I., 1996. Local Polynomial Modelling and Its Applications. Chapman & Hall, London, U.K.

Forde, M., Jacquier, A., Lee, R., 2012. The small-time smile and term structure of implied volatility under the Heston model. SIAM Journal on Financial Mathematics 3 (1), 690–708.

Fouque, J.-P., Papanicolaou, G., Sircar, K. R., 2000. Derivatives in Financial Markets with Stochastic Volatility. Cambridge University Press, London.

Gao, K., Lee, R., 2014. Asymptotics of implied volatility to arbitrary order. Finance and Stochastics 18 (2), 349–392.

Gatheral, J., Hsu, E. P., Laurence, P., Ouyang, C., Wang, T.-H., 2012. Asymptotics of implied volatility in local volatility models. Mathematical Finance 22 (4), 591–620.

Hagan, P. S., Woodward, D. E., 1999. Equivalent Black volatilities. Applied Mathematical Finance 6, 147–157.

Henry-Labordère, P., 2008. Analysis, Geometry, and Modeling in Finance: Advanced Methods in Option Pricing. CRC Press, Boca Raton, FL.

Heston, S., 1993. A closed-form solution for options with stochastic volatility with applications to bonds and currency options. Review of Financial Studies 6, 327–343.

Jacquier, A., Lorig, M., 2015. From characteristic functions to implied volatility expansions. Advances in Applied Probability 47, 837–857.

Jarrow, R. A., Rudd, A., 1982. Approximate option valuation for arbitrary stochastic processes. Journal of Financial Economics 10, 347–369.

Kristensen, D., Mele, A., 2011. Adding and subtracting Black-Scholes: A new approach to approximating derivative prices in continuous-time models. Journal of Financial Economics 102, 390–415.

Kunitomo, N., Takahashi, A., 2001. The asymptotic expansion approach to the valuation of interest rate contingent claims. Mathematical Finance 11 (1), 117–151.

Kunitomo, N., Takahashi, A., 2003. On validity of the asymptotic expansion approach in contingent claim analysis. The Annals of Applied Probability 13 (3), 914–952.

Ledoit, O., Santa-Clara, P., Yan, S., 2002. Relative pricing of options with stochastic volatility. Tech. rep., University of California at Los Angeles.

Lewis, A. L., 2000. Option Valuation under Stochastic Volatility. Finance Press, Newport Beach, CA.

Li, C., 2014. Closed-form expansion, conditional expectation, and option valuation. Mathematics of Operations Research 39, 487–516.

Medvedev, A., Scaillet, O., 2007. Approximation and calibration of short-term implied volatilities under jump-diffusion stochastic volatility. Review of Financial Studies 20 (2), 427–459.

Takahashi, A., Yamada, T., 2012. An asymptotic expansion with push-down of Malliavin weights. SIAM Journal on Financial Mathematics 3 (1), 95–136.

Xiu, D., 2014. Hermite polynomial based expansion of European option prices. Journal of Econometrics 179, 158–177.

Yoshida, N., 1992. Asymptotic expansions for statistics related to small diffusions. Journal of the Japan Statistical Society 22, 139–159.