Telling from Discrete Data Whether the Underlying Continuous-Time Model Is a Diffusion

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ABSTRACT

Can discretely sampled financial data help us decide which continuous-time models are sensible? Diffusion processes are characterized by the continuity of their sample paths. This cannot be verified from the discrete sample path: Even if the underlying path were continuous, data sampled at discrete times will always appear as a succession of jumps. Instead, I rely on the transition density to determine whether the discontinuities observed are the result of the discreteness of sampling, or rather evidence of genuine jump dynamics for the underlying continuous-time process. I then focus on the implications of this approach for option pricing models.

IN MANY INSTANCES IN FINANCIAL ECONOMETRICS, we make inference about a postulated continuous-time model on the basis of discretely sampled observations. Among potential continuous-time models, most specifications adopted have been diffusions, although the literature is more and more frequently allowing for jumps (see Merton (1976), Ahn and Thompson (1988), Bates (1991), Das and Foresi (1996), Duffie, Pan, and Singleton (2000), Aït-Sahalia, Wang, and Yared (2001), among others).

A diffusion process is a Markov process with continuous sample paths. Suppose we observe the process every Δ units of time, with Δ not necessarily small. Presented with such a discrete subsample of the continuous-time path, can we tell whether the underlying model that gave rise to the data was a diffusion, or should jumps be allowed into the model? Intuition suggests that the answer should be no. After all, the discrete data are purely discontinuous even if the continuous-time sample is not. Thus, faced with two discon-

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tinuous samples, how could we ever rule out that one came from a diffusion but not the other? It turns out that this question is not as hopeless as it first sounds. A finer look reveals that there are different degrees of discontinuity in the discrete observations, some compatible with the continuity of the underlying sample path, some not.

The approach I use relies on identifying a necessary and sufficient restriction on the transition densities of diffusions, at the sampling interval of the observed data. The theory is based on Karlin and McGregor's (1959b) notion of coincidence probabilities combined with crossing arguments, leading to the total positivity restriction. This restriction characterizes the continuity of the unobservable complete sample path and is valid for every sampling interval including long ones.

In a nutshell, the argument is based on the fact that if a diffusion on the real line starts below another diffusion, it cannot finish above the second one without their sample paths having crossed at least once.[1] Since the discrete data reveal the transition density at whatever sampling interval is available, one can actually discriminate between diffusions and nondiffusion Markov processes on the basis of their discrete subsamples.

An essential property of the characterization is that if the transitions over an interval of length Δ satisfy the inequality characterization, then longer transitions will satisfy it as well. That is, we only need to focus on a single criterion, rather than attempt to verify a large number of conditions, to determine whether the process is a diffusion. Furthermore, the criterion can determine whether a discrete transition function is compatible with *some* diffusion without requiring that a list of potential candidate diffusions be exhibited.

To provide some intuition, I give the corresponding version of the criterion for discrete-state models, both for continuous-time Markov chains and discrete-time trees. In a discrete-state world, by definition, all the state changes are jumps, and the appropriate notion of continuity distinguishes between small or continuous jumps, which are those from one state to an immediately adjacent one, from large or discontinuous jumps, which are those from one state to a nonadjacent one. One consequence for trees in option pricing is that binomial and trinomial trees with branches continuously spaced are inappropriate as approximations to jump-diffusions, despite their common use in that context.

Finally, I employ the criterion function to determine whether the risk-neutral transition density of the S&P 500 implied by observed option prices is compatible with an underlying continuous-time diffusion for the index, or

[1] An alternative property is that the eigenvalues of the infinitesimal generator of the diffusion are all real and nonnegative (see Mandl (1968)). This property has been exploited by Florens, Renault, and Touzi (1998). As a practical matter, however, eigenvalues and eigenfunctions of generators can be difficult to calculate. The first eigenvalue and eigenfunctions can be determined only in certain special cases (see, e.g., Aït-Sahalia (1996a) and Hansen, Scheinkman, and Touzi (1998)).

whether jumps should be included. The latter happens to be the case empirically. I examine the consequences of this finding for the implied diffusion, implied tree, and Edgeworth expansion approaches that are widely used in practice to price and hedge equity derivatives. Another possible empirical motivation, which is not pursued here, would be to substantiate, or invalidate, the approach of modeling the dynamics of the short term interest rate, or other factors taken individually, as continuous-time diffusions: should they be diffusions, or something else within the Markov class?

The paper is organized as follows. Section I examines the implications of the continuity of the sample paths for the discrete data and obtains a necessary and sufficient characterization of the transition density of diffusions. Section II interprets this criterion in terms of discrete-state processes. Section III gives two examples. Section IV focuses on the dynamics implicit in S&P 500 option prices. Section V summarizes the results and concludes.

I. Implications of the Continuity of the Sample Paths for the Discrete Data

A. Sample Path-level Characterization of Continuity

I start with some mathematical preliminaries, then establish the notation used throughout the paper. Let us assume that the process $\{r_t, t \geq 0\}$ is Markovian. That is, the continuous-time process $\{r_t, t \geq 0\}$ is defined on a probability space $(\Omega, \Im, P)$ and takes values in $D \subseteq \mathbb{R}$. Further, with $F_t \equiv \sigma(r_s : s \leq t)$, assume that

$$P(r_{t+s} \in I | F_t) = P(r_{t+s} \in I | r_t) \tag{1}$$

for all t, $s \geq 0$, and open interval I in D. This by itself can be an interesting hypothesis to examine (see the companion paper Aït-Sahalia (2000)), but will be maintained throughout this paper. We are then interested in learning, from discrete observations $\{r_0, r_\Delta, r_{2\Delta}, \ldots, r_{n\Delta}\}$, whether the process belongs to the smaller class of diffusion processes. Note that I restrict attention to univariate processes, so the state space is an interval D on the real line.

Let $p(s, y | t, x)$ be the conditional density of $r_s = y$ given $r_t = x$. This is the transition function of the process. To be able to infer the transition function from a time series of observations on r, we must assume that the joint densities of the process are time homogenous. That is, each pair of observations $(r_{t+\Delta},\ r_t)$ at each date t are drawn from the same joint density, which I denote as $p(\Delta, y, x)$ and is independent of t. This assumption lets us in effect transform what would have been a single data point, the observed path, into repeated observations on the pair $(r_{t+\Delta},\ r_t)$ drawn from a common distribution (see Figure 1).

Then let $p(\Delta, y | x)$ denote the transition function of the process over a time interval of length Δ, that is, the conditional density of $r_{t+\Delta} = y$ given $r_t = x$. Without time-homogeneity, the transition function would be $p(t + \Delta, y | t, x)$,

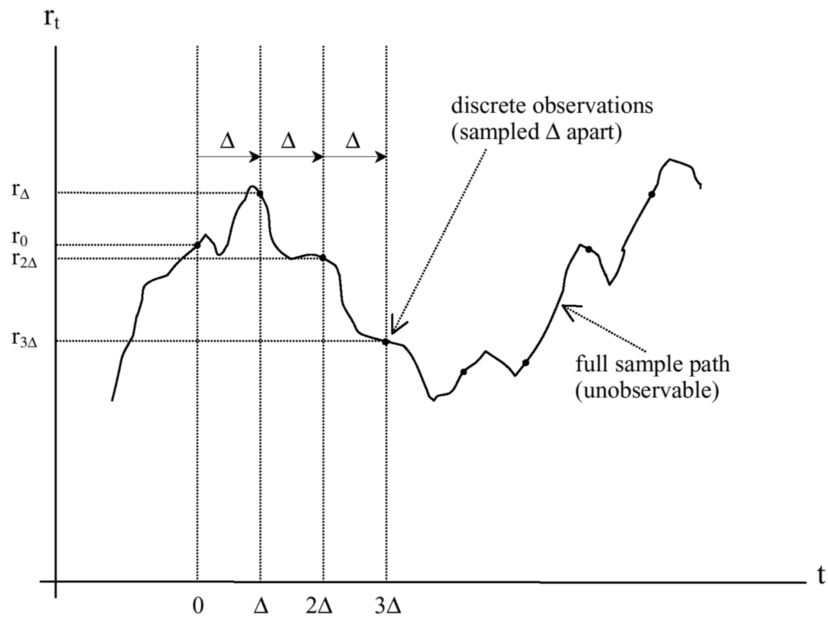


Figure 1. Discrete and continuous sample paths. This figure contrasts the discretely observed sample path with the continuous-time one (which generated the observed sample). It also illustrates the notion of repeated sampling under time homogeneity, with pairs of successive observations $(r_{i\Delta}, r_{(i+1)\Delta})$, sampled Δ units of time apart and drawn from a common joint distribution that depends on Δ but not i.

that is, a function of t and Δ separately. This does not mean that we are only considering stationary processes. Time-homogeneity is necessary for the stationarity of the process (where all finite dimensional distributions are identical no matter where taken in time) but not sufficient. For instance, a Brownian motion is time homogenous, but, of course, not stationary.

I further assume that the sample paths of r are right-continuous with left limits, so the finite-dimensional distributions determine the probabilities of all events, and that r is a strong Markov process. That is, r restarts at the first passage of a given point and retains its transition densities afterwards. More precisely, recall that F_s is the σ-algebra generated by $\{r_t, 0 \leq t \leq s\}$. A stopping time of the process $\{r_t, t \geq 0\}$ is a random variable τ such as the events $\{\tau \leq t\}$ belong to F_t for all $t \geq 0$. Then $\{r_t, t \geq \tau\}$ is a Markov process independent of F_t and has the same transition densities as before τ. A sufficient condition on the transition density p of the process that guarantees that it is a strong Markov process is Feller's property: For every bounded continuous real function f, and for every $t > 0$, $x \mapsto \int_D f(y) p(t, y|x) dy$ defines a continuous function on D (see Ray (1956) and Friedman (1975, Theorem 2.2.4)).

Note also that the transition densities of a Markov process must satisfy the Chapman–Kolmogorov equation:

$$p(2\Delta, y|x) = \int_{\underline{r}}^{\bar{r}} p(\Delta, y|z)p(\Delta, z|x)\,dz \qquad (2)$$

for every x and y in D and $\Delta > 0$. This fact will be handy later.

Then a diffusion is a process satisfying the assumptions above and whose sample paths are continuous everywhere, except possibly for jumps from the boundaries of the state space D.

B. Transition-level Characterization of Continuity

Given that the full path is not observable, the first step in the approach will be to move away from the sample path characterization of diffusions that was just given and examine what can be said about their transitions. Since the process is Markovian and its transitions are time homogenous, the information contained in the discrete data can be summarized through the transition function $p(\Delta, y|x)$. The following example describes what an easy solution to the problem would be, if it were available. Suppose that we find, using any available data analysis technique, that the discrete data can be represented accurately by the conditional density

$$r_{t+\Delta}|r_t \sim N(\gamma_0 + \gamma_1 r_t, \delta_0^2), \qquad (3)$$

that is, a Gaussian transition density with affine mean and constant variance.

Can we then construct a continuous-time diffusion which, based on the empirical evidence (3), could have generated the data as a discrete sample off its continuous sample path? The answer is yes. Consider the diffusion

$$dr_t = \kappa(\alpha - r_t)dt + \sigma dZ_t \qquad (4)$$

for which $p(\Delta, y|x)$ is Gaussian with $E[r_{t+\Delta}|r_t] = r_t + (\alpha - r_t)\exp[-\kappa\Delta]$ and $V[r_{t+\Delta}|r_t] = \sigma^2(1 - \exp[-2\kappa\Delta])/(2\kappa)$. Now set $\kappa = -\ln[1 - \gamma_1]/\Delta$, $\alpha = \gamma_0/(1 - \gamma_1)$ and $\sigma^2 - 2\delta_0^2 \ln[1 - \gamma_1]/[(1 - \gamma_1^2)\Delta]$: the continuous-time diffusion is fully determined. Based on the available data, we cannot rule out (4) as a continuous-time model that could have generated the discrete data.

Unfortunately, such explicit calculations are impossible to conduct in most cases, since, in general, one cannot compute in closed-form the transition function $p(\Delta, y|x)$ implied by a particular diffusion model, or vice versa, although very accurate closed-form approximations can be formed (see Aït-Sahalia (1999, 2002)). On the other hand, this explicit calculation, whenever available, provides a constructive answer to the problem in that not only do we get to answer that a diffusion could have generated the discrete data, but we also get to identify that diffusion.

Alternatively, if we knew what specific diffusion to look for—that is, had an idea as to what functions μ and σ^2 to use in $dr_t = \mu(r_t)dt + \sigma(r_t)dZ_t$—then we could place a restriction on the discrete transitions of the process for any Δ, even without knowing in closed form what its transition density is. As shown in Aït-Sahalia (1996b), time-homogeneity of the transition density can be exploited by rewriting the forward and backward Fokker–Planck–Kolmogorov equations: Note that by stationarity $p(s,y|t,x) = p(s-t,y|0,x) \equiv p(s-t,y|x)$ for any $s > t > 0$ and, therefore, $\partial p/\partial s = -\partial p/\partial t$.

We can then eliminate derivatives of p with respect to $\Delta = s - t$; that is, the left-hand side terms in

$$\begin{cases} \dfrac{\partial p(\Delta,y|x)}{\partial \Delta} = -\dfrac{\partial}{\partial y}(\mu(y)p(\Delta,y|x)) + \dfrac{1}{2}\dfrac{\partial^2}{\partial y^2}(\sigma^2(y)p(\Delta,y|x)) \\ \dfrac{\partial p(\Delta,y|x)}{\partial \Delta} = \mu(x)\dfrac{\partial}{\partial x}(p(\Delta,y|x)) + \dfrac{1}{2}\sigma^2(x)\dfrac{\partial^2}{\partial x^2}(p(\Delta,y|x)) \end{cases} \tag{5}$$

for all x and y in the interior of D since the two right-hand sides must be equal.[2] So the terms that could not be estimated with discrete data, that is, the derivatives $\partial p/\partial \Delta$, are now gone. But if we wish to test whether $p(\Delta,y|x)$ could possibly be the discrete transition function from a diffusion, (5) is of no help, since obviously we cannot examine every possible pair of functions (μ,σ^2), unless by sheer luck we happened to stumble upon the right choice for (μ,σ^2).

C. Implications of Continuity for Small Time Transitions

So, what can be said about the transition densities of a diffusion without precommitting to specific choices of (μ,σ^2)? In other words, does the fact that the underlying model is a diffusion—*any* diffusion—imply that its transition densities will have distinguishing features? I will assume that the transition function of the process satisfies the following: for each fixed $\epsilon > 0$ and x in the interior of D, there exists some $\kappa > 0$ such that

$$\int_{|y-x|>\epsilon} p(\Delta,y|x)\,dy = o(\Delta^{\kappa}), \tag{6}$$

as Δ goes to zero. This is not a costly restriction. The limit of the left-hand side as Δ goes to zero is zero. The additional requirement is that this convergence occur at a polynomial rate (at least). This is satisfied, for instance, by the natural generalization from processes driven by a Brownian motion, that is, diffusions, to processes driven by a Lévy process. A Lévy process, like

[2] The rationale for eliminating the term $\partial p/\partial \Delta$ is that this term cannot be estimated given data sampled at a fixed interval Δ. We can only estimate $p(\Delta|y,x)$, $p(2\Delta|y,x)$, and so forth, and their derivatives with respect to the starting and ending state levels x and y.

a Brownian motion, has stationary and independent increments, but those increments are not necessarily Gaussian, in which case a Lévy process can jump. Specifically, we have the following.

LEMMA 1: *For processes driven by a Lévy process, the term in equation (6) is of order* $O(\Delta)$.

The natural first approach is to examine whether the small time characterizations of continuity, that is, those valid in the limit where Δ goes to zero, extend to any discrete time interval. For instance, it is known that the sample paths of the process r are almost surely continuous functions of t if and only if, for every compact interval $I \subseteq D$ and $\epsilon > 0$, Lindeberg's condition

$$\int_{|y-x|>\epsilon} p(\Delta, y|x)\,dy = o(\Delta) \tag{7}$$

as Δ goes to zero is satisfied uniformly in x on D (see Ray (1956)). This condition maps out the continuity of the sample path into a bound on the size of the probability of leaving a given neighborhood in the amount of time Δ; intuitively, this probability must be small as Δ goes to zero if the sample paths are to remain continuous. Condition (7) says *how small* this probability must be as Δ gets smaller. The answer is $o(\Delta)$; that is, negligible compared to Δ. Condition (6) only says that it is of order $o(\Delta^{\kappa})$ for some $\kappa > 0$. Condition (7) says that we must have $\kappa = 1$ uniformly for the sample paths to be continuous. Combining this with Lemma 1, we must therefore have $\kappa < 1$ for discontinuous Lévy processes.

In terms of deciding whether the discrete data came from a diffusion, condition (7) represents a step forward compared to the notion of continuity of the sample path in that, with the right data, it could be verified statistically. However, it is also clear that this condition still cannot be used as a basis of a test for diffusions. Indeed, unless we are presented with ultra-high-frequency data, we do not have the necessary data to examine the behavior of the transition densities as the sampling interval Δ goes to zero.[3] Condition (7) does not restrict the observable transition function $p(\Delta, y|x)$ for the fixed sampling interval of the dataset.[4] An alternative is to focus on a sufficient characterization for the continuity of sample paths such as Kolmogorov's criterion: If there exist $\beta > 0$ and $\gamma > 0$ such that as Δ goes to zero

$$E[|r_{t+\Delta} - r_t|^{\beta}] = O(\Delta^{1+\gamma}), \tag{8}$$

[3] Testing whether the underlying data-generating process is a diffusion might not be a sensible thing to do with ultra-high-frequency data given market microstructure noise (such as bid-ask bounces). This might be less of an issue with decimalization.

[4] Since the Markov property is assumed, we can imply the shorter transitions $p(\Delta/m, y|x)$, $m \geq 2$, from the longer ones, $p(\Delta, y|x)$ by solving equation (2), that is, solving $p(\Delta, y|x) = \int_{\underline{r}}^{\bar{r}} p(\Delta/2, y|z)p(\Delta/2, z|x)\,dz$ for the function $p(\Delta/2, y|x)$. To base a test on the behavior of $p(\Delta/m, y|x)$ as $m \to \infty$ would necessitate that this difficult numerical task be repeated a number of times.

then the process r (has a version which almost surely) has continuous sample paths (see, e.g., Friedman (1975), Theorem 1.2.2). For the same reason, this criterion cannot be used to form a test either: It only restricts the transitions of the process over infinitesimal instants[5] and does not extend to longer time intervals.

How about checking whether the density is determined by its conditional mean and variance? The Markov process r is entirely determined by the two functions μ and σ^2 defined by the limits

$$\begin{cases} \lim_{\Delta\to 0} \frac{1}{\Delta} \int_{|y-x|<\epsilon} (y-x)p(\Delta, y|x)dy = \mu(x) \\ \lim_{\Delta\to 0} \frac{1}{\Delta} \int_{|y-x|<\epsilon} (y-x)^2 p(\Delta, y|x)dy = \sigma^2(x) \end{cases} \tag{9}$$

uniformly for x in the interior of D (see, e.g., Feller (1971, Section X.4)). But this is equally unhelpful, as it is only true in the limit where Δ goes to zero. If all discrete transitions were characterized by the first two moments, then they would all be Gaussian.

D. Transition-level Characterization of Continuity for Any Sampling Interval

In light of the above discussion, we would like a characterization of diffusions that satisfies the following criteria: (D1) the characterization must not rely on observing very fine transitions in time of the process; (D2) unlike the approaches suggested just above, it must not require that either we already know what candidate drift and diffusion functions (μ, σ^2) to use or that the discrete transition happen to match those of the small set of diffusions for which a closed-form solution is available; (D3) it should be based on a necessary and sufficient characterization of diffusions; (D4) it should require that a single property be checked for the particular sampling interval Δ of the available data; (D5) but it should nevertheless be sufficient to ensure that the property is satisfied for the longer observable time intervals. By that, I mean that if we observe the process at say, the daily frequency, and verify that the property holds at that frequency, then it should automatically be the case that the property is also satisfied at the lower observable frequencies of one observation every two days, or one every three days, and so forth.

So, what works? The approach that I propose to use to discriminate between diffusions and non-diffusions, on the basis of discrete-time information, relies on total positivity of order two property of the transition function of a diffusion (see Karlin and McGregor (1959b)). This approach leads to

[5] For alternative characterizations, all sharing a local character in Δ, see Gikhman and Skorohod (1969, Section IV.5).

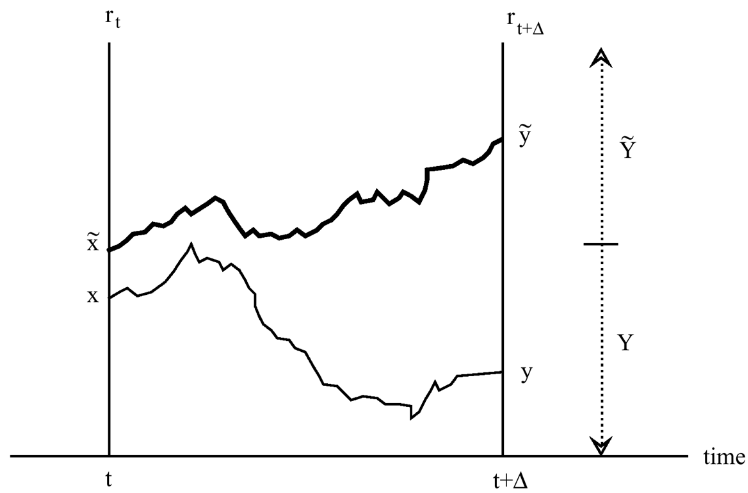


Figure 2. Two sample paths with no crossing. This figure shows the sample paths between two successive sampling dates t and $t + \Delta$ followed by the two processes r (thin curve) and $\tilde{r}$ (thick curve), with r positioned at time t at x, below $\tilde{r}$ which starts at $\tilde{x}$. The processes reach time $t + \Delta$ having maintained the same order throughout, r finishing in the set Y below $\tilde{r}$ in the set $\tilde{Y}$.

identifying and then verifying the necessary and sufficient condition that the function $p(\Delta, y|x)$ must satisfy if it were to correspond to the discrete transition function of a continuous-time diffusion, without ever requiring that we identify in closed form the diffusion. Unless the transition function $p(\Delta, y|x)$ fails to satisfy that condition, after accounting for the sampling noise of the estimator if any, the hypothesis that the discrete observations came from a diffusion will not be rejected.

What follows is a heuristic approach that delivers the main result of this theory. Consider two processes $\{r_t, t \geq 0\}$ and $\{\tilde{r}_t, t \geq 0\}$ on $\mathbb{R}$, having the same transition probability densities p, but otherwise independent. Suppose that $r_t = x$, while $\tilde{r}_t = \tilde{x}$ with $x < \tilde{x}$. The essential consequence of the continuity of sample paths is that, at any future date $t + \Delta$, r cannot be above $\tilde{r}$ without their sample paths having crossed at least once between t and $t + \Delta$. Consider two potential values at $t + \Delta$, $y < \tilde{y}$, and two sets Y and $\tilde{Y}$ such that all values in Y are smaller than those in $\tilde{Y}$. The probability that $r_{t+\Delta} \in Y$ and $\tilde{r}_{t+\Delta} \in \tilde{Y}$, without their sample paths having ever crossed between t and $t + \Delta$, is (see Figure 2):

$$
\begin{aligned}
&\Pr(r_{t+\Delta} \in Y, \tilde{r}_{t+\Delta} \in \tilde{Y}, \{\forall \tau \in [t, t+\Delta], r_\tau \neq \tilde{r}_\tau\} | r_t = x, \tilde{r}_t = \tilde{x}) \\
&\quad = \Pr(r_{t+\Delta} \in Y, \tilde{r}_{t+\Delta} \in \tilde{Y} | r_t = x, \tilde{r}_t = \tilde{x}) \qquad (10) \\
&\qquad - \Pr(r_{t+\Delta} \in Y, \tilde{r}_{t+\Delta} \in \tilde{Y}, \{\exists \tau \in [t, t+\Delta] / r_\tau = \tilde{r}_\tau\} | r_t = x, \tilde{r}_t = \tilde{x}).
\end{aligned}
$$

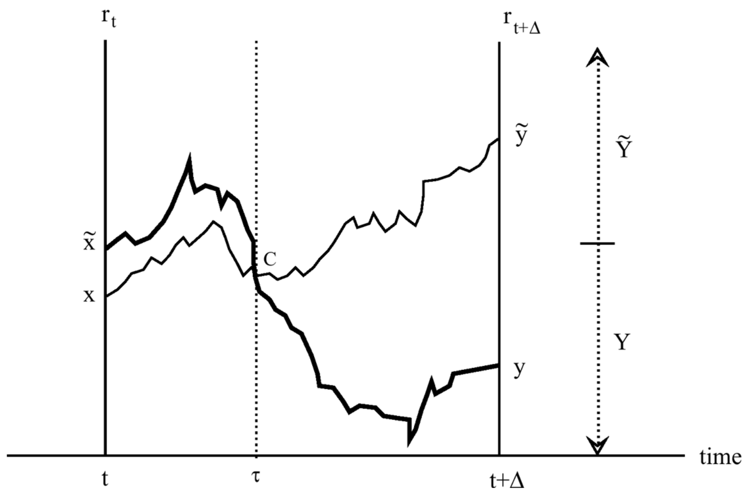


Figure 3. Two sample paths crossing. This figure shows the sample paths between two successive sampling dates t and $t + \Delta$ followed by the two processes r (thin curve) and $\tilde{r}$ (thick curve), with r starting below $\tilde{r}$ at date t but finishing above it at date $t + \Delta$. If the sample paths are continuous, this can only happen if they cross (at least) once in between. In the figure, the coincidence time (where the two processes are in the same state C) is denoted by τ.

To evaluate the second term on the right-hand side, let τ be the first coincidence time between t and $t + \Delta$, that is, the smallest τ such that $r_\tau = \tilde{r}_\tau$ (see Figure 3). Of course, we do not actually observe the time τ, since we do not see what happens between t and $t + \Delta$; we simply know that τ exists as part of the event to be evaluated and that τ is a stopping time by the strong Markov property discussed above.

By the reflection principle and the commonality of distributions, after time τ, we can no longer tell which is r and which is $\tilde{r}$. Therefore, we can interchange them as

$$
\begin{aligned}
&\Pr(r_{t+\Delta} \in Y, \tilde{r}_{t+\Delta} \in \tilde{Y}, \{\exists \tau \in [t, t+\Delta]/r_\tau = \tilde{r}_\tau\} | r_t = x, \tilde{r}_t = \tilde{x}) \\
&\quad = \Pr(\tilde{r}_{t+\Delta} \in Y, r_{t+\Delta} \in \tilde{Y}, \{\exists \tau \in [t, t+\Delta]/r_\tau = \tilde{r}_\tau\} | r_t = x, \tilde{r}_t = \tilde{x}) \qquad (11) \\
&\quad = \Pr(\tilde{r}_{t+\Delta} \in Y, r_{t+\Delta} \in \tilde{Y} | r_t = x, \tilde{r}_t = \tilde{x}),
\end{aligned}
$$

where the last equality follows since the sample paths r and $\tilde{r}$ must have crossed between t and $t + \Delta$, for r, having started below $\tilde{r}$ at t, to finish above it at $t + \Delta$. This fact intrinsically characterizes the continuity of the sample paths. For instance, if the process can have jumps, r and $\tilde{r}$ may reverse order without ever crossing (see Figure 4).

Incidentally, Figures 3 and 4 illustrate why the argument is inherently univariate. If the process can evolve in three dimensions as opposed to being restricted to the plane, then the two replications r and $\tilde{r}$ can interchange

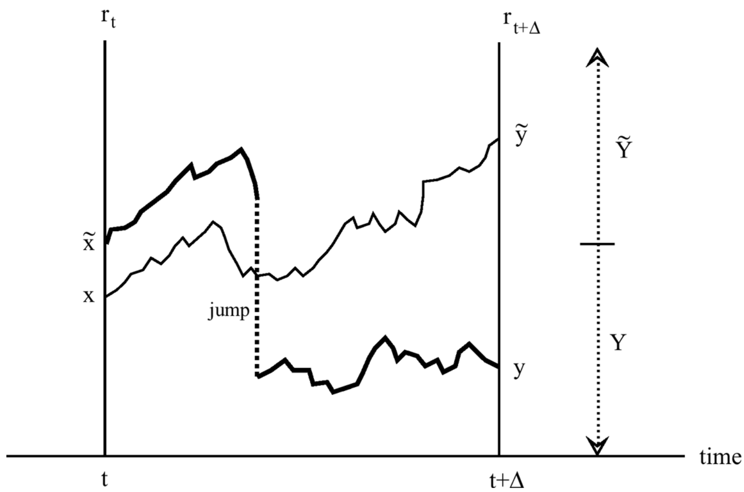


Figure 4. Two sample paths with jumps. This figure shows the sample paths between two successive sampling dates t and $t + \Delta$ followed by the two processes r and $\tilde{r}$, with r starting above $\tilde{r}$ at date t but finishing below it at date $t + \Delta$. By contrast with the situation illustrated in Figure 3, if the sample paths can jump, this reversal of order may happen without the two processes having ever been coincident, that is, having occupied the same state at the same time. Note that in both Figure 3 and Figure 4, we do not observe what actually happens between t and $t + \Delta$ (we only see the process every Δ units of time). I will draw inference about what happened in between from the Δ-apart sample, specifically from features of the transition density $p(\Delta, y|x)$.

order without jumping and without ever crossing as they do in Figure 3. Just imagine in Figure 4 that, instead of representing a jump on the plane, the dotted part of the path of $\tilde{r}$ represents a bridge that goes continuously over, or under, the path of r. In that situation, the two paths of r and $\tilde{r}$ remain continuous throughout, yet the two processes have reversed order without crossing.

Back now to the univariate case, by independence of r and $\tilde{r}$, we have

$$\begin{cases} \Pr(r_{t+\Delta} \in Y, \tilde{r}_{t+\Delta} \in \tilde{Y} | r_t = x, \tilde{r}_t = \tilde{x}) = P(\Delta, Y|x) P(\Delta, \tilde{Y}|\tilde{x}) \\ \Pr(\tilde{r}_{t+\Delta} \in Y, r_{t+\Delta} \in \tilde{Y} | r_t = x, \tilde{r}_t = \tilde{x}) = P(\Delta, Y|\tilde{x}) P(\Delta, \tilde{Y}|x), \end{cases} \tag{12}$$

where $P(\Delta, Y|x) \equiv \int_{y \in Y} p(\Delta, y|x)\,dx$ denotes the common cumulative distribution function of the two processes.

Consequently the probability that $r_{t+\Delta} \in Y$ and $\tilde{r}_{t+\Delta} \in \tilde{Y}$, without their sample paths having ever crossed between t and $t + \Delta$, is

$$P(\Delta, Y|x) P(\Delta, \tilde{Y}|\tilde{x}) - P(\Delta, Y|\tilde{x}) P(\Delta, \tilde{Y}|x) > 0. \tag{13}$$

The inequality follows from the fact that the probability of any possible event is positive. For the inequality to be strict, assume that every transition is possible in the sense that $P(\Delta, Y|x) > 0$ for every x in D, $Y \subset D$, and $\Delta > 0$, that is, the process is strict. Otherwise, replace strictly greater by greater or equal to in inequality (13). Intuitively, this inequality states that the probability that the relative ranking of r and $\tilde{r}$ remain unchanged, between t and $t + \Delta$, is greater than the probability that their ranking be reversed. If the process is a diffusion, then inequality (13) must be satisfied for every $x < \tilde{x}$ and $Y < \tilde{Y}$ in the state space.

Assuming that the density function $p(\Delta, y|x)$ is continuous in y for each x, it follows from inequality (13) that the transition function of any diffusion process must obey the inequality

$$\delta(\Delta, y, \tilde{y}|x, \tilde{x}) \equiv p(\Delta, y|x)p(\Delta, \tilde{y}|\tilde{x})x - p(\Delta, y|\tilde{x})p(\Delta, \tilde{y}|x) > 0 \tag{14}$$

for any $x < \tilde{x}$ and $y < \tilde{y}$ in D.

E. Properties of the Transition-level Characterization of Continuity

I now verify that the total positivity characterization (14) of diffusions satisfies the criteria that I had set earlier to be the basis of a discriminating criterion for discretely sampled diffusions. Firstly, inequality (14) is valid for any Δ, not just infinitesimally small ones, so (D1) is satisfied. Secondly, (D2) is obviously verified as well: Unlike condition (5), inequality (14) makes no reference to the unknown μ and σ^2 functions of the diffusion process. I now check that this inequality is not only necessary, but also sufficient to characterize a diffusion process.

PROPOSITION 1: *If its transition densities satisfy the inequality (14) for every* $\Delta > 0$*, then the process is a diffusion.*

Therefore, if inequality (14) holds for all $\Delta > 0$, then the process r has (almost surely) continuous sample paths. Hence the characterization of diffusions (14) satisfies (D3). Determining whether the discrete data could have come from a diffusion will be based on checking property (14), or its equivalent forms (13) or (15) below, for the sampling interval Δ corresponding to that of the data, thereby satisfying (D4). All that remains to be proved is that (D5) is satisfied, that is, the following proposition.

PROPOSITION 2: *If inequality (14) holds for the sampling interval* Δ*, then it must hold for longer observable time intervals as well.*

Thus all five requirements are satisfied. Finally, condition (14) can be expressed in even simpler terms when the transition function is smooth.

PROPOSITION 3: *Assume that* $p(\Delta, y|x)$ *is strictly positive and twice-continuously differentiable on* $D \times D$*. Then (14) is equivalent to*

$$\frac{\partial^2}{\partial x \partial y} \ln(p(\Delta, y|x)) > 0 \quad \text{for all } \Delta > 0 \text{ and } (x, y) \in D \times D. \tag{15}$$

In the rest of the paper, I will refer to inequality (15) as the "diffusion criterion." Given a transition function $p(\Delta, y|x)$, satisfying (15) is equivalent to the proposition that the underlying continuous-time model that gave rise to those discrete transitions was a diffusion. For a given Δ, finding a pair of (x, y) where the criterion fails is sufficient (absent sampling noise) to reject the hypothesis that the underlying model could have been a diffusion.

One final remark. We know that if X is a diffusion, then any deterministic, twice-continuously differentiable and strictly monotonic function of X will also be a diffusion.[6] It would be desirable for the diffusion criterion to also satisfy this invariance property. This is indeed the case. Namely, we have the following proposition.

PROPOSITION 4: *Criterion (15) is invariant with respect to Itô transformations of the process.*

II. Interpretation: Discrete-state Processes

A. Continuous-time, Discrete-state Markov Chains

To help interpret condition (14), or equivalently (15), and understand its implications for trees in derivative pricing, consider a continuous-time, stationary, Markov chain that can only take countable discrete values, say, $\{\ldots,-1,0,1,\ldots\}$. When does such a process have continuous sample paths? Obviously, the notion of continuity of a sample path depends on the state space: in $\mathbb{R}$, this is the usual definition of a continuous function. More generally, by *continuity*, one means continuity with respect to the order topology of the state space of the process. In a discrete-state space, the appropriate notion of continuity of the chain's sample paths is the following intuitive one: It constrains the chain to never jump by more than one state at a time, either up or down. It turns out that the restriction on the chain's transition probabilities analogous to (14) characterizes precisely this form of continuity.

Specifically, assume that the Markov chain is right-continuous and let $J_0, J_1, \ldots$ be the jump times of the chain $X = \{X_t\}_{t\geq 0}$, defined by

$$J_0 = 0, \qquad J_{n+1} = \inf\{t \geq J_n / X_t \neq X_{J_n}\} \tag{16}$$

for $n = 0,1,\ldots$, with the convention that $\inf\{\varnothing\} = \infty$ (see Figure 5). Consider now the restriction for the transition matrix of the Markov chain that is analogous to condition (14) for the conditional density of a diffusion, namely:

$$\begin{aligned}\delta(\Delta, j, \tilde{j}|i, \tilde{i}) \equiv\ & \Pr(X_{t+\Delta} = \tilde{j}|X_t = \tilde{i})\Pr(X_{t+\Delta} = j|X_t = i) \\ & - \Pr(X_{t+\Delta} = \tilde{j}|X_t = i)\Pr(X_{t+\Delta} = j|X_t = \tilde{i}) \geq 0\end{aligned} \tag{17}$$

[6] Such transformations are commonly used in finance, for instance, to go from an arithmetic Brownian motion to a geometric Brownian motion, from a square-root process to a Bessel process, from a CEV process (with geometric mean) to a Bessel process, and so forth. Indeed, most closed-form solutions we rely on are obtained through such a transformation.

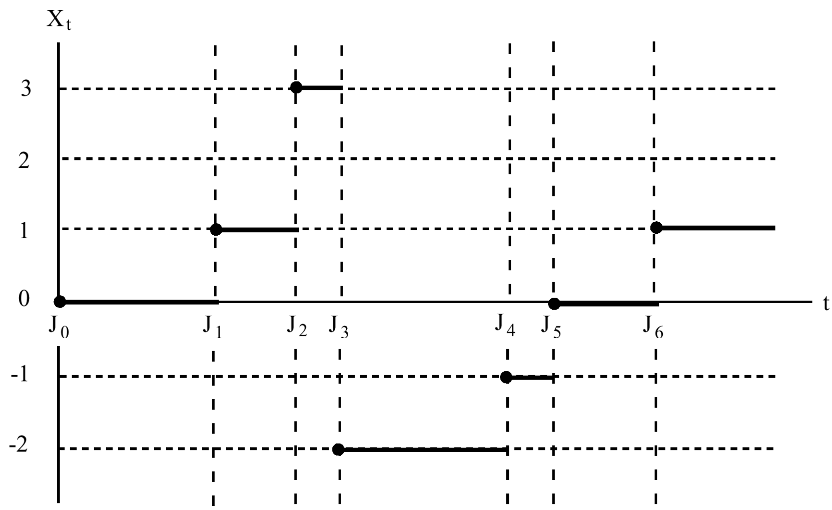


Figure 5. Discrete-time Markov chain. This figure illustrates the concept of continuity of sample paths for a continuous-time, discrete-state Markov chain. By definition, all the state changes are "jumps" and the appropriate notion of continuity distinguishes between "small" or "continuous" jumps, which are those from one state to an immediately adjacent one ($+1$ or -1), from "large" or "discontinuous" jumps, which are those from one state to a nonadjacent one. The jumps occurring at jump times J_1, J_4, J_5, and J_6 are all of size $+1$ or -1, so that the process jumps from one state to an immediately adjacent one. By contrast, the jump taking place at jump times J_2 and J_3 are of size $+2$ and -5, respectively. The first set is compatible with continuity of the sample paths whereas the second set is not.

for all quadruplets of states such that $i < \tilde{i}$ and $j < \tilde{j}$, and all real $\Delta > 0$ (by stationarity, the probabilities above are independent of t). The inequality is strict if we further assume that $\Pr(X_{t+\Delta} = j | X_t = i) > 0$ for every pair of states (i, j) and every $\Delta > 0$. Then we have the following proposition.[7]

Proposition 5: *Condition (17) is equivalent to the restriction that X can only jump from a given state to one of the two immediately adjacent states.*

That is, for every state i, there exists $0 \leq \lambda_i \leq 1$ such that:[8]

$$\Pr(Y_{n+1} = j | Y_n = i) = \begin{cases} \lambda_i & \text{if } j = i + 1 \\ 1 - \lambda_i & \text{if } j = i - 1 \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

where $Y_n \equiv X_{J_n}$. For instance, the example in Figure 5 violates the continuity condition (17) at jump times J_2 and J_3.

[7] See Karlin and McGregor (1959a) for birth and death Markov chains.

[8] Note that λ_i is independent of n by stationarity.

B. Discrete-time, Discrete-state Trees

If we not only discretize the state space but also discretize the time dimension, then the natural representation of the dynamics of the process takes the form of a tree. Let Y_n in this case denote the state of the process after n moves. In full generality, a tree is multinomial, so that if $Y_n = i$ is the state after n moves, then Y_{n+1} can take any one of the possible states. As in the continuous-time Markov chain with transitions described by equation (18), continuity now means that the only possible moves occur to the immediately adjacent states or, in tree parlance, nodes. That is, binomial and trinomial trees are the natural approximation of a diffusion since by construction they restrict moves to take place to the immediately adjacent nodes.

Conversely, binomial and trinomial trees with branches spaced with the same order of magnitude *cannot* approximate discontinuous processes such as jump-diffusions and more general Lévy processes, despite the fact that they are commonly used in practice to price derivatives written on assets with discontinuous price dynamics. Basically, a jump is a move by more than one state at a time and, from what precedes, allowing for the possibility of jumps requires nonzero probabilities of moves to nonadjacent nodes (see Figure 6). I will explore below the implications of this principle for option pricing models based on binomial trees.

III. Some Examples

For now, I return to the continuous-state case of diffusions and give a few examples illustrating the applicability of the criterion function (15) to discriminate between diffusion and nondiffusion continuous-time models on the basis of their discrete-time transition functions.

A. Example 1: Brownian Motion versus Cauchy Jump Process

The simplest possible illustration of the applicability of criterion (15) to distinguish a diffusion from a nondiffusion Markov process is provided by contrasting a Brownian motion, which has continuous sample paths, with a Cauchy process, which jumps.[9] The Brownian motion density

$$p(\Delta, y|x) = (2\pi\sigma^2\Delta)^{-1/2}\exp\{-(y-x)^2/(2\sigma^2\Delta\} \tag{19}$$

satisfies (15), since $\partial^2 \ln(p(\Delta, y|x))/\partial x \partial y = 1/(\sigma^2\Delta) > 0$. However, the Cauchy density

$$p(\Delta, y|x) = (\Delta/\pi)/((y-x)^2 + \Delta^2) \tag{20}$$

[9] While both models have time-homogenous transition densities, neither model is stationary.

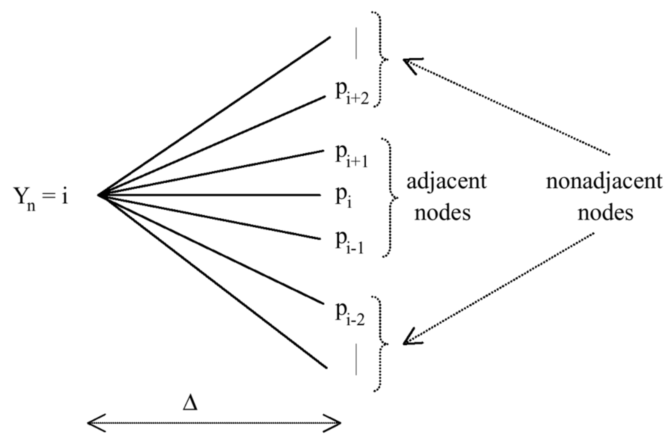


Figure 6. Discrete-time, discrete-state multinomial tree. This figure illustrates the concept of continuity of sample paths for a discrete-time, discrete-state multinomial tree. For the approximated continuous-time, continuous-state process to be Markovian, the tree must be recombining. For the approximated process to be a diffusion, the tree must lead to continuous sample paths, which, in this case, means that all the discrete-time jumps along the tree must occur from one node to an immediately adjacent state. If at date $n\Delta$ the process is in state i, then for the tree to be an approximation to a continuous-path process, it must be that the only nonzero branch probabilities are p_{i-1}, p_i, and p_{i+1}. Since binomial and trinomial trees only have adjacent nodes, these by construction can only approximate a continuous-time, continuous-state process with continuous sample paths, that is, a diffusion. Conversely, for the tree to approximate a process with discontinuous sample paths, some of the nonadjacent nodes must be attainable.

does not satisfy (15)—nor, consequently, (7)—since

$$\frac{\partial^2 \ln(p(\Delta, y|x))}{\partial x \partial y} = 2\left(\frac{\Delta^2 - (y-x)^2}{\Delta^2 + (y-x)^2}\right) < 0 \tag{21}$$

for y sufficiently far apart from x: $|y - x| > \Delta$. However, being Markov processes, they of course both satisfy the Chapman–Kolmogorov equation (2). They also satisfy the conservation requirement (6). Figure 7 plots the transition densities and the diffusion criterion (15) for these two models.

B. Example 2: Variance Gamma Lévy Process

This example illustrates that the approach applies to all nondiffusion processes—not just the usual Poisson jumps, but also more general Lévy jump processes. The Variance Gamma Lévy process is obtained by evaluating a Brownian motion at a random time with Gamma distribution. The result is a pure jump Lévy process. To create a model for stock prices, let $S_t = S_0 \exp(\mu t + VG_t)$ where $VG_t = BM_{\gamma_t}$. BM_t is an arithmetic Brownian motion with drift θ and diffusion σ^2, and γ_t is a random variable with a

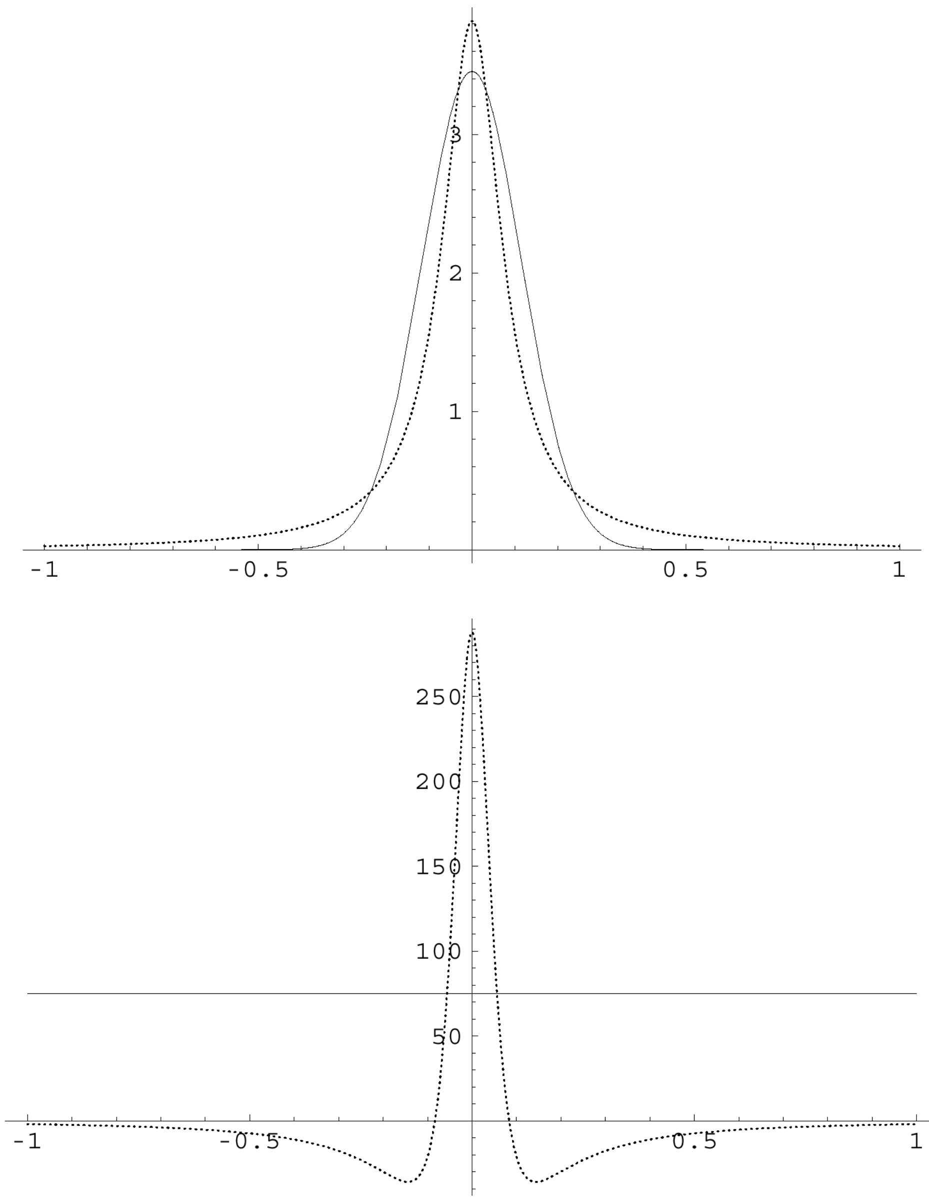


Figure 7. Diffusion and jump process transition functions. The top plot in this figure represents the transition densities of a Brownian motion (solid curve) and a Cauchy process (dotted curve) as a function of the difference $z = y - x$ between the forward and backward state values. The fact that the tails of the Cauchy distribution are larger is apparent. The bottom plot graphs the criterion function $\partial^2 \ln(p(\Delta, y|x))/\partial x \partial y$ for both distributions. While the criterion is always positive for the Brownian motion (solid curve), it is negative in the tails (where z is large in absolute value) for the Cauchy distribution (dashed curve).

Gamma distribution with mean one and variance ν. The transition density of $X_t = \ln(S_t)$ given X_0 is given by (see Madan, Carr, and Chang (1998))

$$p(\Delta, y|x) = \frac{2\exp(z(y|x)\theta/\sigma^2)}{\sqrt{2\pi}\sigma\nu^{\Delta/\nu}\Gamma(\Delta/\nu)}\left(\frac{z(y|x)^2}{\theta^2+2\sigma^2/\nu}\right)^{(\Delta/(1/2\nu)-1/4)} \times BesselK_{(\Delta/\nu-1/2)}\left(\frac{\sqrt{z(y|x)^2(\theta^2+2\sigma^2/\nu)}}{\sigma^2}\right) \tag{22}$$

where $z(y|x) = y - x - \mu\Delta$ and $BesselK_\eta$ is the modified Bessel function of the second kind of order η. Of course, this density violates the diffusion criterion.

IV. The Implied Model from Option Prices: Is It a Diffusion?

A. The Transition Density Implicit in Option Prices

I now turn to the empirical question of determining whether the dynamic model for the underlying asset that is implied by observed option prices could be a diffusion. Suppose that we are interested in pricing at date zero a derivative security written on a traded underlying asset with price process $\{X_t | t \geq 0\}$ and with payoff function $\Psi(X_\Delta)$ at some future date Δ. Let us assume for simplicity that the riskless rate r and the dividend yield δ paid by the asset are constant. It is well known that when markets are dynamically complete, the only price of the derivative security that is compatible with the absence of arbitrage opportunities is

$$P_0 = e^{-r\Delta}E[\Psi(X_\Delta)|X_0 = x_0] = e^{-r\Delta}\int_0^{+\infty}\Psi(x)p(\Delta, x|x_0)\,dx, \tag{23}$$

where $p(\Delta, x|x_0)$ is the transition function (or risk-neutral density, or state-price density) induced by the dynamics of the underlying asset price.

Throughout this section, whenever I refer to the implied dynamics, or the implied model, I refer to their risk-neutral version. Note, however, that because the risk-neutral and actual probability measures assign zero probability to the same events (they have the same null sets), the underlying asset does not jump under one set of probabilities if and only if it does not jump under the other. So if we do not reject the null hypothesis that the underlying asset is a diffusion under the risk-neutral measure, then it must also be a diffusion under the actual probability measure. Conversely, if we reject it under the risk-neutral measure, then the underlying asset must be allowed to jump under the actual probability measure as well.

The Black–Scholes option pricing formula is the prime example of equation (23), when the underlying asset is a diffusion with $\sigma(x) = \sigma x$, with σ constant. The corresponding transition density is the lognormal density

$$P_{BS}(\Delta, x|x_0) = \frac{1}{\sqrt{2\pi\Delta}\sigma x} \exp\{-(\ln(x/x_0) - (r - \delta - \sigma^2/2)\Delta)^2/(2\sigma^2\Delta)\} \tag{24}$$

and so the integral in equation (23) can be evaluated explicitly for specific payoff functions, such as the call option's $\Psi(X_\Delta) = \max(0, X_\Delta - K)$ for a fixed strike price K:

$$H_{BS}(\Delta, K, x_0, \sigma) = e^{-r\Delta}\{F_\Delta \Phi(d_1) - K\Phi(d_2)\}, \tag{25}$$

where $F_\Delta = x_0 \exp\{(r - \delta)\Delta\}$ is the forward price for delivery of the underlying asset at date Δ and $d_1 = (\ln(K/x_0) + (\sigma^2/2)\Delta)/(\sigma\Delta^{1/2})$, $d_2 = d_1 - \sigma\Delta^{1/2}$.

At this point, the common practice when pricing and hedging equity options is to describe the market prices of call options for a given maturity Δ as given by the parametric equation (25), except that the volatility parameter for that maturity is a smooth function $\sigma_{IMP}(K/F_\Delta)$ of the option's moneyness $M = K/F_\Delta$:

$$H(\Delta, K, x_0) = H_{BS}(\Delta, K, x_0, \sigma_{IMP}(K/F_\Delta)). \tag{26}$$

The function $\sigma_{IMP}(K/F_\Delta)$ is often known as the "implied volatility smile." A direct differentiation of the basic no-arbitrage pricing equation (23) with respect to the strike price yields the corresponding risk neutral density.[10]

In the present setup, the only transition function compatible with the observed option prices H must be

$$\begin{aligned} p(\Delta, K|x_0) &= e^{r\Delta} \frac{\partial^2}{\partial K^2} H(\Delta, K, x_0) \\ &= e^{r\Delta} \frac{\partial^2}{\partial K^2} H_{BS}(\Delta, K, x_0, \sigma_{IMP}(K/F_\Delta)) \\ &= e^{r\Delta} \left\{ \frac{\partial^2 H_{BS}}{\partial K^2} + \frac{2}{F_\Delta} \frac{d\sigma_{IMP}}{dM} \frac{\partial^2 H_{BS}}{\partial K \partial \sigma} \right. \\ &\qquad \left. + \frac{1}{F_\Delta^2} \left(\frac{d\sigma_{IMP}}{dM} \right)^2 \frac{\partial^2 H_{BS}}{\partial \sigma^2} + \frac{1}{F_\Delta^2} \frac{d^2\sigma_{IMP}}{dM^2} \frac{\partial H_{BS}}{\partial \sigma} \right\}. \end{aligned} \tag{27}$$

[10] See Banz and Miller (1978), Breeden and Litzenberger (1978), and Aït-Sahalia and Lo (1998) for a nonparametric version.

B. Telling Whether the Implied Dynamic Model Is a Diffusion

I now turn to an empirical implementation of formula (27) with option price data. Option prices, or equivalently their implied volatilities, give us the function H in equation (26). Then equation (27) gives us the transition function for one maturity Δ implicit in the cross-section of option prices at one point in time. I will then check whether this transition function is compatible with an underlying diffusion model for the asset price by applying criterion (15) to that implied transition function.

The data come from the Chicago Board Options Exchange (CBOE) and represent price quotes for call and put options on the Standard & Poor's 500 Index (SPX). The options are European, and to illustrate the methodology, I will focus on a single randomly chosen trading day, March 19, 2001. I repeated the experiment on different days, drawn from different time periods, to insure the robustness of the findings; the results are similar. I report the results for the most complete cross-section of traded strikes that day, the June 2001 expiration. Table I contains the full data set used in the empirical application.

The raw data present three challenges. First, future dividends are not observable; second, S&P 500 futures are traded on the Chicago Mercantile Exchange and cannot easily be time-stamped synchronously with the options to obtain F_Δ; and third, there are often substantial differences in the traded volume and open interest in the call and put with the same strike and maturity, except near the money where both are usually very liquid. I solve the first problem by relying on the spot-forward parity relationship under which the left-hand side of equation (26) depends on the dividend yield δ only through F_Δ. To solve the second problem, I use prices of at-the-money options, where both the put and call are liquid, to infer the value of the implied futures F_Δ according to put-call parity:

$$F_\Delta = K + e^{-r\Delta}\{H(\Delta,K,x_0) - G(\Delta,K,x_0)\}, \tag{28}$$

where G denotes the put price and K is the strike closest to being at the money. Note that this equation does not require that the spot price of the index be recorded; it simply requires the market prices of the at-the-money call and put.

Given the futures price F_Δ, I then replace the prices of all illiquid options, with the price implied by put-call parity applied at each value of the strike price, using the price of the more liquid option. For instance, if the put is more liquid, then the call price is inferred from that of the put with the same strike as $H(\Delta,K,x_0) = G(\Delta,K,x_0) + e^{r\Delta}(F_\Delta - K)$ (this equation is instead solved for G given H when the call is more liquid). This solves the third problem. After this procedure, all the information contained in liquid put prices has been extracted and resides in corresponding call prices. I can now concentrate exclusively on call options.

Table I

SPX Option Data

These options are European calls on the S&P 500 index with prices recorded on March 19, 2001, at 10:30 a.m. CST. For each option's price, I use the bid-ask midpoint. The riskless rate is r = 5.50% (which, following market convention, is slightly higher than the three-month T-bill rate, reflecting the fact that the T-bill rate is not the relevant riskless rate faced by traders). The options expire on June 15, 2001. With the calendar convention, these options have $\Delta = 88$ days to expiration. The at-the-money implied forward price of the index for that maturity is F_Δ = 1,162.93, while the value of the index itself is X_0 = 1,151.10. Time calculations are performed with a 365-day calendar. The moneyness of an option with strike K is $M = K/F_\Delta$.

Strike	Call Price	Moneyness	Implied Volatility
750	408.50	0.6449	0.4131
800	359.90	0.6879	0.3903
850	311.60	0.7309	0.3638
900	264.40	0.7739	0.3463
950	218.40	0.8169	0.3274
995	178.30	0.8556	0.3081
1,025	153.00	0.8814	0.2978
1,050	132.70	0.9029	0.2885
1,075	113.60	0.9244	0.2805
1,100	95.30	0.9459	0.2706
1,125	78.60	0.9674	0.2624
1,130	75.30	0.9717	0.2601
1,140	69.30	0.9803	0.2575
1,150	63.20	0.9889	0.2533
1,160	57.60	0.9975	0.2502
1,170	52.30	1.0061	0.2473
1,175	49.60	1.0104	0.2451
1,180	47.50	1.0147	0.2453
1,190	42.50	1.0233	0.2412
1,200	37.80	1.0319	0.2372
1,210	33.60	1.0405	0.2341
1,225	28.00	1.0534	0.2303
1,250	20.35	1.0749	0.2253
1,275	14.05	1.0964	0.2190
1,300	9.50	1.1179	0.2144
1,325	6.10	1.1394	0.2092
1,350	3.85	1.1609	0.2057
1,375	2.40	1.1824	0.2033
1,400	1.475	1.2039	0.2017
1,425	0.875	1.2254	0.1999
1,450	0.475	1.2469	0.1968
1,475	0.350	1.2684	0.2019
1,500	0.225	1.2898	0.2035

The first step consists in estimating the implied volatility function $\sigma_{IMP}(K/F_\Delta)$ in equation (26). The data reveal quite clearly that an appropriate model for the implied volatility smile is a simple third order polynomial:

$$\sigma_{IMP}(K/F_\Delta) = \beta_0 + \beta_1(K/F_\Delta) + \beta_2(K/F_\Delta)^2 + \beta_3(K/F_\Delta)^3. \tag{29}$$

Table II

Fitted Implied Volatility Smile

This table reports the results of fitting model (29) to the implied volatility data given in Table I. The variable M denotes the option's moneyness $M = K/F_\Delta$. The R^2 of the regression is 0.9993; the adjusted R^2 is 0.9992. The fitted implied volatility function $\sigma_{IMP}(K/F_\Delta)$ is plotted in Figure 8.

Variable	Coefficient	t-statistic	p-value
1	0.4775	10.02	$6 \cdot 10^{-11}$
M	0.5221	3.42	$2 \cdot 10^{-3}$
M^2	−1.3714	−8.59	$2 \cdot 10^{-9}$
M^3	0.6208	11.39	$3 \cdot 10^{-12}$

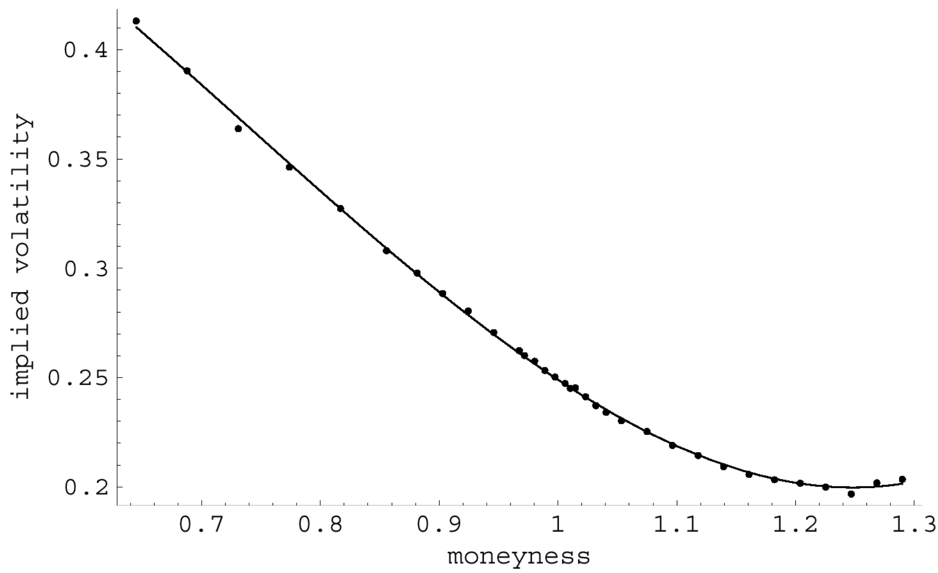
In fact, Table II shows that this model fits the data so well ($R^2 = 0.99$!) that one cannot help but wonder whether self-fulfilling prophecies are at play here. The same is true of other trading days and maturities in this market: The estimated parameters change, but not the quality of the fit.

These options do indeed appear to "trade on a curve"—the curve which is displayed in the top part of Figure 8. In the same way that in the 1970s and pre-1987 1980s the Black–Scholes model in its standard constant volatility form was a reasonably well-accepted paradigm, the current pricing model appears to be well-represented by equation (29) or slight variations of it, including, for instance, slightly different definitions of moneyness. The bottom plot in Figure 8 reports the estimated transition function from equation (27) corresponding to the implied volatility smile in the top plot.

I then apply diffusion criterion (15) to test whether the underlying continuous-time model that produced the observed discrete-time transition represented by the function $p(\Delta, K|x_0)$ could have been a diffusion. Figure 9 displays the criterion function $\partial^2 \ln(p(\Delta, K|x_0))/\partial x_0 \partial K$, which gives the answer: The criterion function is negative in places; hence, the underlying model for the S&P 500 index cannot be a diffusion.

Figure 10 extends the one-day analysis conducted above to every trading day from January 2 until July 27, 2001. The results confirm that there exists no diffusion that would have been able to generate the observed data during that time period.

Could the sampling noise, introduced by the fact that we need to estimate the parameters $\{\sigma_i | i = 0, \ldots, 3\}$ in equation (29), be sufficient to overturn the rejection of the diffusion hypothesis? The fact that model (29) describes the data in an accurate yet parsimonious way suggests that this is unlikely, but let us verify this formally. The effect of the estimation of the implied volatility parameters in equation (29) on the accuracy of the transition density estimator resulting from equation (27) can be assessed by the delta method. The parameter vector β is estimated using a sample of size n from the regression $\sigma = \sigma_{IMP}(K/F_\Delta) + \epsilon$ where ϵ is white noise with variance s^2. The



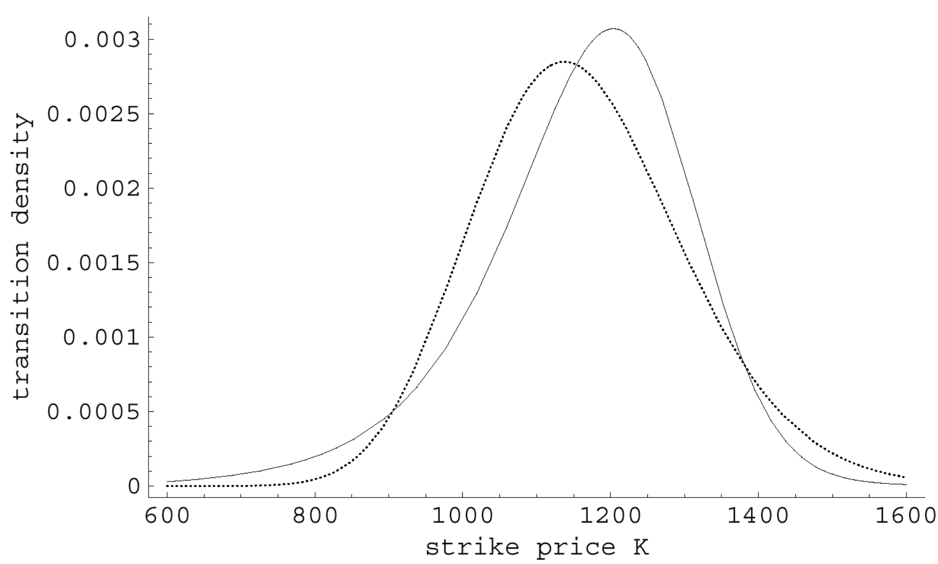


Figure 8. SPX-implied volatility smile and state-price density, March 19, 2001. The top plot in this figure reports the fitted implied volatility function $K/F_\Delta \mapsto \sigma_{IMP}(K/F_\Delta)$. The dots represent the actual implied volatilities from Table I. The parameter estimates are reported in Table II. The solid curve bottom plot in the figure represents the implied transition density $K \mapsto p(\Delta, K|x_0)$ for the maturity Δ and current index value x_0 described in Table I. For comparison purposes, the Black–Scholes state-price density (dotted curve) evaluated at the at-the-money implied volatility is also included. The skewness of the implied density is apparent. All by itself, however, skewness is not a sufficient indication of nondiffusion behavior.

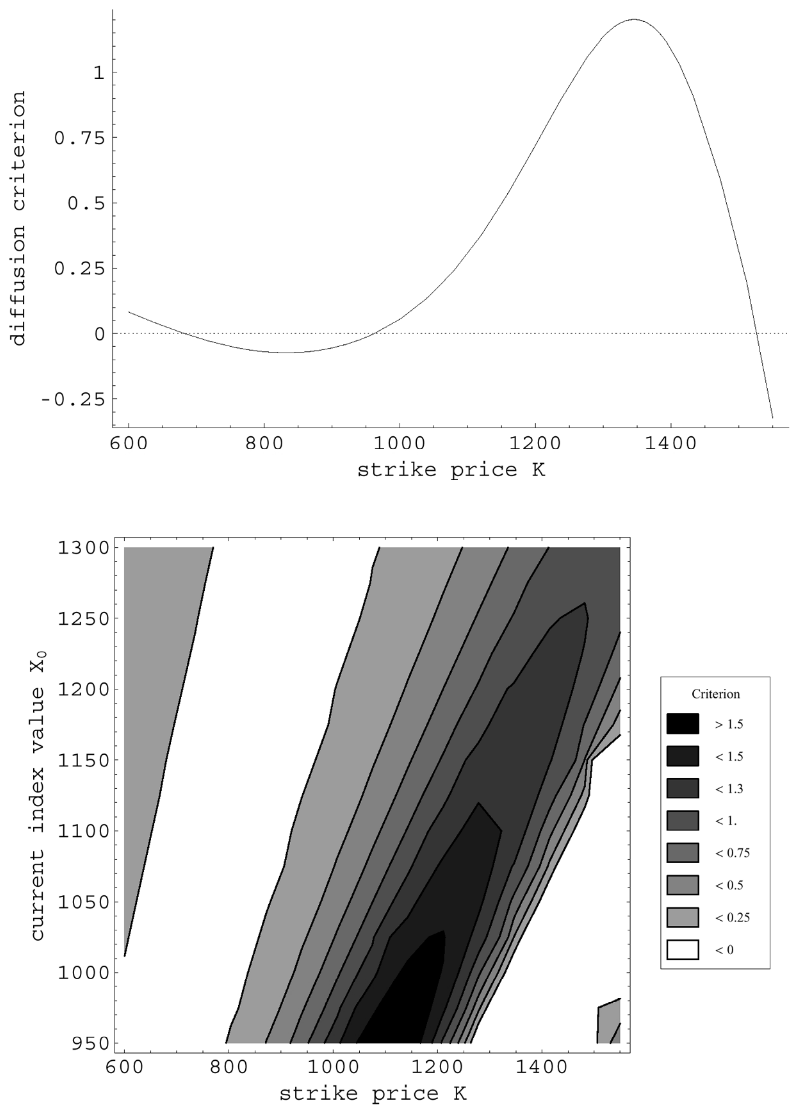


Figure 9. Diffusion criterion applied to the SPX state-price density, March 19, 2001. The two plots in this figure determine whether the option-implied transition density is compatible with a diffusion model for the underlying (risk-neutral) asset price dynamics. The top plot graphs the criterion function $K \mapsto \partial^2 \ln(p(\Delta, K|x_0))/\partial x_0 \partial K$ for the fixed Δ and x_0 corresponding to the option data that are given in Table I. For display purposes, the criterion function is multiplied by the fixed constant 10^4 (this is irrelevant to the conclusions, since we only care about the sign of the criterion). Notice that there are regions where the criterion function becomes negative. The bottom plot is a contour plot, for the fixed Δ in the data, of the three-dimensional surface $(K, x_0) \mapsto \partial^2 \ln(p(\Delta, K|x_0))/\partial x_0 \partial K$. White areas in the contour plot indicate regions where the criterion function is negative. Anything short of nonnegativity of the criterion function for all values of (K, x_0) is incompatible with an underlying diffusion model. The conclusion from the analysis is that there exists no diffusion model that could have generated the transition density (reported in Figure 8) which prices these options.

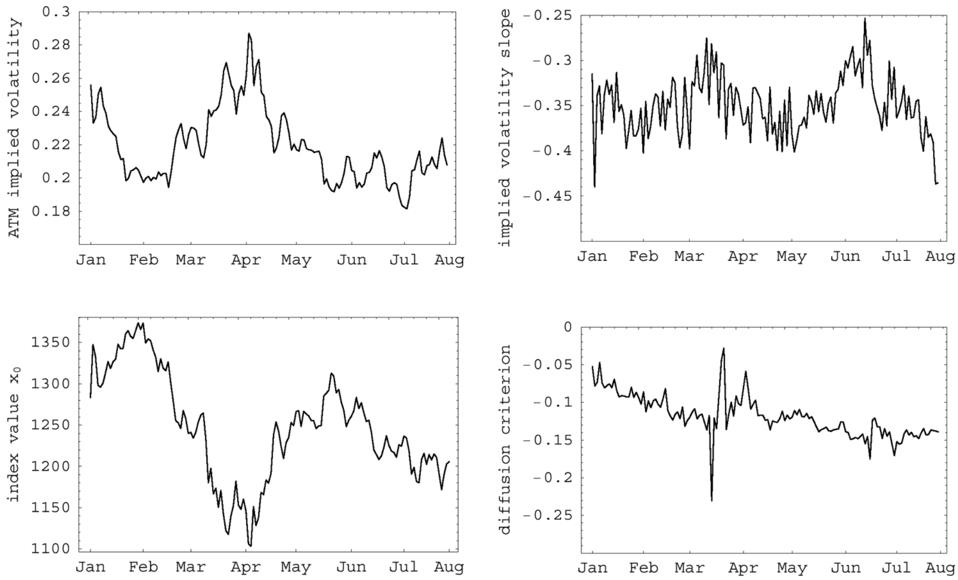


Figure 10. SPX-implied volatility smiles and diffusion criterion, January–July 2001. The four plots in this figure determine whether the option-implied transition density is compatible with a diffusion model for the underlying (risk-neutral) asset price dynamics. The results are obtained by repeating the analysis of Figure 9 for each trading day from January 2 until July 27, 2001. The plots correspond to a constant option maturity of three months each day. Since the maturities traded on a given day vary, a bivariate implied volatility smile model is fitted to the market data each day and is then evaluated at the three month maturity. The actual model is $\sigma_{IMP}(K/F_\Delta,\tau) = \beta_0 + \beta_1(K/F_\Delta) + \beta_2(K/F_\Delta)^2 + \beta_3\tau + \beta_4(K/F_\Delta)\tau$. Unlike the one-day model (29) corresponding to a given traded maturity, no cubic term in moneyness is included in the implied volatility smile model to avoid excessive variability when interpolating implied volatilities across maturities. The resulting at-the-money ($K/F_\Delta = 1.0$) implied volatilities and at-the-money slope of the smile (evaluated between $K/F_\Delta = 0.9$ and 1.1) are reported in the top two graphs. The lower left graph plots the index value x_0 observed that day. Finally, the lower right graph reports the minimum value of the diffusion criterion function $K \mapsto \partial^2 \ln(p(\Delta,K|x_0))/\partial x_0 \partial K$ for the strike level K ranging from $F_\Delta - 4R$ to $F_\Delta + 3R$, for fixed Δ = three months and x_0 corresponding to the value that day. R is the fitted at-the-money standard deviation recorded that day times the square root of three months. Since at its lowest value the criterion function is negative, the conclusion from this more extensive analysis is identical to that from the March 19 data reported in Figure 9.

distribution of the parameter estimates $\hat\beta$ is $n^{1/2}(\hat\beta - \beta) \to N(0,V_\beta)$, where $V_\beta = s^2(M'M)^{-1}$ with M denoting the moneyness K/F_Δ. From this, it follows that

$$n^{1/2}\left(\frac{\partial^2 \ln(\hat p(\Delta,K|x_0))}{\partial K \partial x_0} - \frac{\partial^2 \ln(p(\Delta,K|x_0))}{\partial K \partial x_0}\right) \xrightarrow[n\to\infty]{d} N(0,\nabla'\cdot V_\beta \cdot \nabla)$$

where ∇ denotes the gradient of $\partial^2 \ln(p(\Delta,K|x_0))/\partial x_0 \partial K$ with respect to the parameter vector β, and ∇' the transposed vector. ∇ is easily calculated from equation (27).

A formal test can be based on calculating the minimum value reached by the criterion function $\partial^2 \ln(p(\Delta,K|x_0))/\partial x_0 \partial K$ over the interval of traded strikes $(\underline{K},\overline{K})$. Define this minimum, as a function of the parameters of the implied volatility smile, to be $\lambda(\beta)$. Let $\kappa(\beta)$ be the strike level at which this minimum is reached, that is, the solution of the first order condition for the minimization of the criterion function. The minimum value is $\lambda(\beta) \equiv \partial^2 \ln(p(\Delta,\kappa(\beta)|x_0))/\partial x_0 \partial K$. Keep differentiating each operation with respect to β to get to the gradient $\nabla\lambda$ of the minimum $\lambda(\beta)$ with respect to β. Then again by the delta method,

$$n^{1/2}(\lambda(\hat{\beta}) - \lambda(\beta)) \xrightarrow[n\to\infty]{d} N(0,\nabla\lambda'\cdot V_\beta\cdot\nabla\lambda). \tag{30}$$

Given estimates $\hat{\beta}$ and $\hat{V}_\beta$ of the implied volatility smile, we calculate $\hat{\lambda} \equiv \lambda(\hat{\beta})$ (negative, otherwise there is no rejection) and $\hat{V}_\lambda \equiv \nabla\lambda'\cdot\hat{V}_\beta\cdot\nabla\lambda$ by evaluating the gradient at $\hat{\beta}$. Then the probability that the diffusion null hypothesis is true in the one-sided test of $H_0: \lambda(\beta) \geq 0$ against $H_1: \lambda(\beta) < 0$ is given by

$$\text{Prob}(H_0\, true|\hat{\lambda},\hat{V}_\lambda) = \text{Prob}(\lambda > 0|\hat{\lambda},\hat{V}_\lambda) = 1 - \Phi(-n^{1/2}\hat{V}_\lambda^{-1/2}\hat{\lambda}), \tag{31}$$

where Φ is the standard Normal cumulative distribution function.

Not surprisingly, the empirical result is that this noise is insufficient by itself to account for the negativity of the criterion function, that is, the probability (31) is essentially zero, with the minimum value $\hat{\lambda}$ being negative and approximately 10 times larger than its standard error $n^{-1/2}\hat{V}_\lambda^{1/2}$. The rejection of the diffusion hypothesis follows.[11] Given the quality of the fit of the implied volatilities, it is not surprising that the sampling noise is not sufficient to overturn the rejection, that is, to make the negativity of $\partial^2 \ln(p(\Delta,K|x_0))/\partial x_0 \partial K$ become statistically insignificant.

One last remark. Recall that the test for diffusion relies on a one-factor assumption for the underlying asset returns. Could it be that rejecting that the underlying asset returns came from a diffusion is due to that univariate maintained hypothesis? Suppose that the model for the underlying asset price involves two factors, X as before and now Y. For concreteness, think of Y as representing either another asset or X's stochastic volatility. Let

[11] The situation would be quite different if the implied volatility model (29) were nonparametric, since we would then be subjected to the curse of differentiation in equation (27), and the sampling noise would be substantially greater. But given the fit of the basic parametric model, there seems to be no need for additional flexibility when modeling implied volatilities, at least with S&P 500 index options during that time period.

$p(\Delta, x, y | x_0, y_0)$ be the corresponding transition density. The price of a derivative contract with payoff dependent on X_Δ is

$$\begin{aligned}
P_0 &= e^{-r\Delta} E[\Psi(X_\Delta) | X_0 = x_0, Y_0 = y_0] \\
&= e^{-r\Delta} \int_{-\infty}^{+\infty} \int_0^{+\infty} \Psi(x) p(\Delta, x, y | x_0, y_0) \, dx \, dy \\
&= e^{-r\Delta} \int_{-\infty}^{+\infty} \int_0^{+\infty} \Psi(x) \left\{ \int_{-\infty}^{+\infty} p(\Delta, x, y | x_0, y_0) \, dy \right\} dx \\
&= e^{-r\Delta} \int_0^{+\infty} \Psi(x) p(\Delta, x | x_0, y_0) \, dx,
\end{aligned}$$

where now $p(\Delta, x | x_0, y_0) \equiv \int_{-\infty}^{+\infty} p(\Delta, x, y | x_0, y_0) \, dy$ represents the marginal in x from the conditional density of (x, y) given (x_0, y_0). Since the conditioning information is irrelevant in equation (27), the density extracted by the method of equation (27) should therefore be $p(\Delta, x | x_0, y_0)$ instead of $p(\Delta, x | x_0)$.

Hence, if there were an additional factor, it should appear in the conditioning set of that density. For that, it should appear in the pricing formula H. Given that H is given by equation (26), the only place an additional factor could enter would be in the function σ_{IMP}, that is, in the market-driven implied volatility smile. So to the extent that the market prices options using a deterministic smile model (the function σ_{IMP} which does not depend upon a second factor) and the evidence suggests that this is an accurate description of the reality (recall that $R^2 = 0.99$ for model (29)), the transition density implied by the market data will be a function of (x, x_0) only, not (x, x_0, y_0). By contrast, it is clear that trying to model the evolution in time of the transition density would undoubtedly require additional factors, and stochastic volatility can usefully play the role of the missing latent variable. (See Ait-Sahalia (2000) for the role of latent variables such as stochastic volatility in tests of the Markov hypothesis.)

C. Consequences for Approximations of the Dynamics of the Underlying Asset and Implied Models

Dupire (1994) showed that, if the underlying model is a diffusion, then the call pricing function $H_t(\Delta, K, x_0)$ for maturity Δ at instant t necessarily satisfies the following forward form of the no-arbitrage pricing partial differential equation:

$$\frac{\partial H_t}{\partial \Delta} = -\{r - \delta\} K \frac{\partial H_t}{\partial K} + \frac{1}{2} \frac{\partial^2 H_t}{\partial K^2} \sigma(K, t)^2 K^2 - \delta H_t, \tag{32}$$

from which it follows immediately that the volatility function compatible with the call pricing function is

$$\sigma^2(K, t) = \left\{ \frac{\partial H_t}{\partial \Delta} + \{r - \delta\} K \frac{\partial H_t}{\partial K} + \delta H_t \right\} \Big/ \left\{ \frac{1}{2} \frac{\partial^2 H_t}{\partial K^2} K^2 \right\}. \tag{33}$$

Dumas, Fleming, and Whaley (1998) used this methodology to empirically test different specifications of the function σ^2 against option data.

The analysis of the preceding section shows that, in light of the options data, there exists no such implied volatility function no matter how general it is allowed to be, since the underlying model is not a diffusion. In other words, the right-hand side of (33) will not produce a function of (K, t) only. An appropriate modification of equation (32) allowing for discontinuities in the sample paths of the underlying asset would have to include difference terms (in addition to, or instead of, the differential terms).

Derman and Kani (1994) and Rubinstein (1994) propose binomial and trinomial tree approximations to the risk-neutral dynamics of the underlying asset that, by construction, replicate the observed option prices. Combining the empirical result of this section—that the underlying model is not a diffusion—with the characterization of tree approximations for continuous path processes in Section II.B, the conclusion is that these implied trees should be extended to more than three path possibilities at each node in order to be approximate discontinuous price paths. Otherwise, their continuous-time limit is constrained to be a continuous path process, which we have seen is not the case empirically.

Therefore, contrary to the common practice in derivative pricing, jump-diffusion processes should be approximated by multinomial trees with more than three branches at least at some of their nodes. Of course, a binomial or trinomial tree could approximate a jump process but only if at least one of the branches leads to a discrete change in the asset value, that is, a change of order one. This is not usually the case when trees are used in practice: The branches are often equally spaced, or at least the price changes from one node to the next over a time interval of length Δ are continuous in magnitude, that is, of order $\sqrt{\Delta}$.

D. Structural Alternatives to Implied Volatility Smiles

Modeling option prices by an implied volatility smile is inherently a purely descriptive approach. It is a very accurate description of actual market prices, but nevertheless remains a reduced-form approach. One of the main structural alternatives to the implied smile approach consists in extending the Black–Scholes in a variety of possible directions, which all involve relaxing the Normality assumption for the underlying asset returns that is built into equation (24) for the price density.

Consider first the ad hoc Edgeworth expansions that have been proposed in the literature.[12] These expansions replace the Normal density for returns with an Edgeworth expansion that allows for excess skewness and kurtosis in risk-neutral asset returns. If the stock price is S, the riskless rate r, σ the standard deviation of the stock returns, and μ_3 and μ_4 denote the standard-

[12] See for example Jarrow and Rudd (1982).

ized skewness and kurtosis, respectively, these expansions for the log-returns are typically in the form

$$p(\Delta, x|x_0) = \frac{\exp\{-z^2/2\}}{\sqrt{2\pi}}\left(1 + \frac{\mu_3}{6}(z^3 - 3z) + \frac{\mu_4}{24}(z^4 - 6z^2 + 3)\right), \qquad (34)$$

where

$$z = z(\Delta, x|x_0) = \frac{x - x_0 - (r - \sigma^2/2)\Delta}{\sigma\sqrt{\Delta}},$$

with $x = \ln(S_\Delta)$ and $x_0 = \ln(S_0)$. Given equation (34), option pricing formulas can be obtained by applying equation (23) with the density p for prices replaced by its implication from the returns density (34).

Note that the transition density $p(\Delta, x|x_0)$ in this model is a function of (x, x_0) only through $z = x - x_0$. Note that, beyond option pricing, this form of space homogeneity has important statistical consequences when combined with time homogeneity. Even if the model is not stationary, its first differences are, thereby making the analysis of maximum-likelihood estimators substantially simpler. Indeed, treating the first observation as fixed, the likelihood function is

$$\ell_n(\theta) \equiv \sum_{i=1}^{n} \ln\{p_\theta(\Delta, r_{i\Delta}|r_{(i-1)\Delta})\} = \sum_{i=1}^{n} \ln\{p_\theta(\Delta, r_{i\Delta} - r_{(i-1)\Delta})\} \qquad (35)$$

and in the latter form involves only the data $(r_{i\Delta} - r_{(i-1)\Delta})$.

But could equation (34) represent the transition density of a diffusion? In other words, we need to find out what are the diffusions with space-homogeneous transition functions. Using the diffusion criterion, it turns out that the class of such processes is rather small.[13]

PROPOSITION 6: *The only diffusion with a space-homogenous transition function is the (arithmetic) Brownian motion, that is, the process with $\sigma(x) = \sigma$ and $\mu(x) = \mu$ both constant.*

The Edgeworth expansion (34) is in the space-homogenous class $p(\Delta, x|x_0) = q(\Delta, x - x_0)$. But from Proposition 6, the Gaussian distribution is the only diffusion process with a space-homogeneous transition function $q(\Delta, x - x_0)$. Hence, there is no diffusion model for the underlying stock returns that can be represented by these ad hoc expansions, other than the Gaussian density

[13] The result can be obtained differently by calculating the Laplace transform of a Polya frequency density of order two (see Karlin (1968, Chapter 7) for definitions and Theorem 5.2).

for which $\mu_3 = \mu_4 = 0$. Note that this is not just saying that equation (34) is not, in general, a proper density. There is just no density, other than the Gaussian, that can represent a diffusion and be space homogeneous.[14]

This leads quite naturally to the next idea. How about maintaining the Gaussian assumption but making the conditional mean and variance more complex than in the Black–Scholes model? Could that accommodate a diffusion process? Unfortunately, the answer is, here too, negative. Suppose that we restrict attention to processes with Gaussian distributions and use the results here to prove differently a well-known result regarding Markov processes with Gaussian transitions. Namely, we have the following proposition.

PROPOSITION 7: *The Ornstein–Uhlenbeck process,*

$$dX_t = (\alpha - \beta X_t)dt + \sigma dZ_t, \tag{36}$$

is not only the only diffusion with a Gaussian transition function, but also the only such Markov process.

The conditional mean and variance of the process are affine and constant respectively, $E[X_{t+\Delta}|X_t = x] = \alpha/\beta + (x - \alpha/\beta)e^{-\beta\Delta}$ and $V[X_{t+\Delta}|X_t = x] = \sigma^2(1 - e^{-2\beta\Delta})/2\beta$, which reduce to $E[X_{t+\Delta}|X_t = x] = x$ and $V[X_{t+\Delta}|X_t = x] = \sigma^2\Delta$ if $\beta = 0$ (the arithmetic Brownian motion special case). The implication of this for potential structural extensions of the Black–Scholes model wishing to remain within the diffusion class is that, in the same fashion that the model cannot be extended to non-Gaussian but space-homogenous densities, it cannot be extended either to Gaussian densities with conditional mean and variances that are more complex than affine and constant, respectively. The latter extension would, in fact, take us even outside of the Markov class, in which case pricing via risk-neutral expectations conditioned on the current asset price given by equation (23) no longer holds.

V. Conclusions

Within the Markov world, diffusion processes are characterized by the continuity of their sample paths. When looking at discrete data, are the discontinuities observed the result of discreteness, or are they the result of nondiffusion behavior on the part of the underlying continuous-time data-generating process? This paper examines the implications for the discrete data of having been generated by a univariate diffusion, on the basis of a criterion that uniquely characterizes the transition densities of diffusions

[14] Since the transition density of the Variance Gamma Lévy process is space homogeneous but not Gaussian, it cannot represent the transition function of a diffusion. However, the function $p(\Delta, K|x_0)$ produced by equation (27) is not a function of $K - x_0$. Hence, Proposition 6 cannot be applied to rule out immediately the possibility of an underlying diffusion model when using an implied volatility smile. We must use diffusion criterion (15) to be able to tell.

and is equivalent to continuity of the continuous-time, unobservable, sample paths. It relies solely on the transition function, an object that can be inferred from the discrete observations.

I also interpreted this characterization in terms of discrete-state processes, first continuous-time Markov chains and then discrete-time trees. The intuitive result is that the characterization in this case means that the process can only jump by one state at a time. I then drew some implications for the approximations used in derivative pricing, depending upon whether the underlying model is or is not a diffusion. Finally, I tested whether the underlying model for the asset price dynamics that is implied by S&P 500 option prices could have been a diffusion and spelled out some implications for the implied diffusion, implied tree, and Edgeworth expansions approaches to option pricing.

One final remark. From a discretely sampled time-series $\{r_0, r_\Delta, r_{2\Delta}, \ldots, r_{n\Delta}\}$, one could test nonparametrically the hypothesis that the data were generated by a continuous-time diffusion $\{r_t, t \geq 0\}$. Formally,

$$\begin{cases} H_0 : \partial^2 \ln(p(\Delta, y|x))/\partial x \partial y > 0 & \text{for all } x, y \\ H_1 : \partial^2 \ln(p(\Delta, y|x))/\partial x \partial y \leq 0 & \text{for some } x, y, \end{cases}$$

and one could base a test on checking whether (15) holds for a nonparametric estimator of the density $p(\Delta, y|x)$. Locally polynomial estimators can be used for that purpose (see Aït-Sahalia (2000)). Their use in the present context is left to future work.

Appendix

Proof of Lemma 1: Decompose the Lévy process into the sum of three independent Lévy processes, $X^{(1)}$, $X^{(2)}$, and $X^{(3)}$ where $X^{(1)}$ is a linear transform of a Brownian motion with drift, $X^{(2)}$ is a compound Poisson process having only jumps of size at least one and $X^{(3)}$ is a pure-jump martingale having only jumps of size at most one (see, e.g., Theorem I.42 in Protter (1990)). For $X = X^{(1)} + X^{(2)} + X^{(3)}$ to change by at least ϵ in time Δ, it must be that at least one of the three components changes by at least $\epsilon/3$. Thus, the result needs only be proved for each component individually. For the Brownian part $X^{(1)}$, we have of course $\Pr(|X_\Delta^{(1)} - X_0^{(1)}| > \epsilon | X_0^{(1)}) = o(\Delta)$. For the "big jump" part $X^{(2)}$, the probability of one jump occurring is $O(\Delta)$, while more than one jump occurs with probability $o(\Delta)$. Hence, we also have $\Pr(|X_\Delta^{(2)} - X_0^{(2)}| > \epsilon | X_0^{(2)}) = O(\Delta)$.

That leaves the "small jump" part $X^{(3)}$. For that pure-jump martingale part M, we have first by Chebyshev's Inequality

$$\Pr(|M_\Delta| > \epsilon) \leq \epsilon^{-2} E[(M_\Delta)^2];$$

then by Burkholder's Inequality (see, e.g., Theorem IV.54 in Protter (1990)),

$$E[(M_\Delta)^2] \leq C_2 E[[M_\Delta, M_\Delta]],$$

where C_2 is a constant and $[M_\Delta, M_\Delta] \equiv \sum_{0\leq s\leq\Delta}(\Delta M_s)^2$ is the martingale's quadratic variation with $\Delta M_s \equiv X_s - X_{s-}$ denoting the jump magnitude of M at time s. Finally, we have

$$E\left[\sum_{0\leq s\leq\Delta}(\Delta M_s)^2\right] = \Delta\int_{-1}^{+1} x^2 v(dx)$$

from the Corollary following Theorem I.38 in Protter (1995); ν is the Lévy measure, which always satisfies $\int_{-\infty}^{+\infty}\min(x^2,1)v(dx) < \infty$. Thus for the "small jump" component $X^{(3)}$ we have $\Pr(|X_\Delta^{(3)} - X_0^{(3)}| > \epsilon | X_0^{(3)}) = O(\Delta)$, too. Q.E.D.

Proof of Proposition 1: In this proof, I follow the original argument of Karlin and McGregor (1959b). To prove the claim, it suffices to show that (14) implies that the convergence in (7) is uniform in x, which is known to be a necessary and sufficient condition for the continuity of the sample paths. Let I be a compact interval included in the domain $D = (\underline{r}, \bar{r})$ of the process r. That (7) holds for every fixed $\epsilon > 0$ and fixed x in I is fairly innocuous. Indeed, the limit (6) occurs at a polynomial rate, that is

$$P(\Delta, (\underline{r}, x-\epsilon) \cup (x+\epsilon, \bar{r}) | x) = \int_{|y-x|>\epsilon} p(\Delta, y|x)dy \leq c_{\epsilon,x}\Delta^\kappa \tag{37}$$

for some constants $c_{\epsilon,x} > 0$ and $\kappa > 0$ (κ is not necessarily greater or equal to one). Then it follows from (13) that

$$\begin{aligned} &P(\Delta, (x+\epsilon, \bar{r})|x)P(\Delta, (x+\epsilon/4, x+3\epsilon/4)|x+\epsilon/2) \\ &\quad \leq P(\Delta, (x+\epsilon/4, x+3\epsilon/4)|x)P(\Delta, (x+\epsilon, \bar{r})|x+\epsilon/2). \end{aligned} \tag{38}$$

From (37), we have that

$$\begin{cases} P(\Delta, (x+\epsilon/4, x+3\epsilon/4)|x) \leq c_{\epsilon/4,x}\Delta^\kappa \\ P(\Delta, (x+\epsilon, \bar{r})|x+\epsilon/2) \leq c_{\epsilon/2,x+\epsilon/2}\Delta^\kappa \end{cases}$$

and, since from (6) there exists $\zeta > 0$ such that $P(\Delta, (x+\epsilon/4, x+3\epsilon/4)|x+\epsilon/2) \geq 1/2$ for all $0 \leq \Delta \leq \zeta$, it follows from (38) that $P(\Delta, (x+\epsilon, \bar{r})|x) \leq 2d_{x,\epsilon}\Delta^{2\kappa}$ for some constant $d_{\epsilon,x} > 0$. This process can be repeated an arbitrary number of times, each time multiplying the constant κ by a factor of two. Therefore, the constant κ can be replaced by one, or for that matter any number, but one will be sufficient in what follows.

The crucial aspect now is to prove the uniformity of the convergence in x. Fix $\epsilon > 0$ and let $x_0 < x_1 < \ldots < x_m$ be a finite partition of I, such that $x_i - x_{i-1} = \epsilon/2$ for $i = 1, \ldots, m$. Fix $\tilde{\epsilon} > 0$. From what precedes with κ now replaced by one, for each $i = 1, \ldots, m$, there exists $\xi_i > 0$ such that for all Δ, $0 \leq \Delta \leq \xi_i$ implies $P(\Delta, (x_i + \epsilon/2, \bar{r}) | x_i) \leq (\tilde{\epsilon}/2)\Delta$. For x in I, consider the particular x_i such that $x - \epsilon/2 < x_{i-1} \leq x < x_i < x + \epsilon/2$. For $0 \leq \Delta \leq \xi_i$, it follows from (13) that

$$\begin{aligned} P(\Delta, (x+\epsilon, \bar{r})|x) P(\Delta, (\underline{r}, x+\epsilon)|x_i &\leq P(\Delta, (x+\epsilon, \bar{r})|x_i) P(\Delta, (\underline{r}, x+\epsilon)|x) \\ &\leq P(\Delta, (x+\epsilon, \bar{r})|x_i) \\ &\leq P(\Delta, (x_i + \epsilon/2, \bar{r})|x_i) \\ &\leq (\tilde{\epsilon}/2)\Delta. \end{aligned} \tag{39}$$

Now, from (6), there exists $\zeta_i > 0$ such that $P(\Delta, (\underline{r}, x_{i-1} + \epsilon)|x_i) \geq 1/2$ for all $0 \leq \Delta \leq \zeta_i$. Next, for Δ satisfying $0 \leq \Delta \leq \inf\{\xi_i, \zeta_i\}$, we have that $P(\Delta, (x + \epsilon, \bar{r})|x) \leq \tilde{\epsilon}\Delta$ since $P(\Delta, (\underline{r}, x + \epsilon)|x_i) \geq P(\Delta, (\underline{r}, x_{i-1} + \epsilon)|x_i)$. Now define $\eta_I \equiv \inf\{\{\xi_i, \zeta_i\} | i = 1, \ldots, m\}$, which is independent of the particular x in I. Thus, there exists $\eta_I > 0$ such that for any $0 \leq \Delta \leq \eta_I$, $P(\Delta, (x + \epsilon, \bar{r})|x) \leq \tilde{\epsilon}\Delta$ for all x in I. Similarly, there exists $\chi_I > 0$ such that for any $0 \leq \Delta \leq \chi_I$, $P(\Delta, (\underline{r}, x - \epsilon)|x) \leq \tilde{\epsilon}\Delta$ for all x in I. Since

$$\int_{|y-x|>\epsilon} p(\Delta, y|x)\, dy = P(\Delta, (\underline{r}, x-\epsilon)|x) + P(\Delta, (x+\epsilon, \bar{r})|x) \leq 2\tilde{\epsilon}\Delta,$$

it follows that the convergence in (7) holds uniformly over x in the compact interval I. Q.E.D.

Proof of Proposition 2: First recall that diffusions are Markov processes; hence, they must satisfy the Chapman–Kolmogorov equation (2). Therefore,

$$\begin{aligned} \delta(2\Delta, y, \tilde{y}|x, \tilde{x}) &= \int_{\underline{r}}^{\bar{r}} \int_{\underline{r}}^{\bar{r}} p(\Delta, y|z) p(\Delta, z|x) p(\Delta, \tilde{y}|\tilde{z}) p(\Delta, \tilde{z}|\tilde{x})\, dz\, d\tilde{z} \\ &\quad - \int_{\underline{r}}^{\bar{r}} \int_{\underline{r}}^{\bar{r}} p(\Delta, y|\tilde{z}) p(\Delta, \tilde{z}|\tilde{x}) p(\Delta, \tilde{y}|z) p(\Delta, z|x)\, dz\, d\tilde{z} \\ &= \int_{\underline{r}}^{\bar{r}} \int_{\underline{r}}^{\bar{r}} \delta(\Delta, y, \tilde{y}|z, \tilde{z}) p(\Delta, z|x) p(\Delta, \tilde{z}|\tilde{x})\, dz\, d\tilde{z} \\ &= \int_{\underline{r}}^{\bar{r}} \int_{\underline{r}, z<\tilde{z}}^{\bar{r}} \delta(\Delta, y, \tilde{y}|z, \tilde{z}) [p(\Delta, z|x) p(\Delta, \tilde{z}|\tilde{x}) \\ &\quad - p(\Delta, \tilde{z}|x) p(\Delta, z|\tilde{x})]\, dz\, d\tilde{z}, \end{aligned}$$

where I have exploited the symmetry property that $\delta(\Delta, y, \tilde{y}|\tilde{z}, z) = -\delta(\Delta, y, \tilde{y}|z, \tilde{z})$ for any $(z, \tilde{z})$, so we only need to integrate over the half-quadrant where $z < \tilde{z}$. As a result,

$$\delta(2\Delta, y, \tilde{y}|x, \tilde{x}) = \int_{\underline{r}}^{\bar{r}} \int_{\underline{r}, z<\tilde{z}}^{\bar{r}} \delta(\Delta, y, \tilde{y}|z, \tilde{z})\delta(\Delta, z, \tilde{z}|x, \tilde{x})\, dz\, d\tilde{z} > 0 \tag{40}$$

since (14) holds for all transitions of length Δ. The same argument shows that the property is satisfied for all integer multiples of Δ, that is, for all observable frequencies that can be deducted from the highest available frequency $1/\Delta$. Q.E.D.

Proof of Proposition 3: Consider $x < \tilde{x}$ and $y < \tilde{y}$. Rearranging inequality (14) yields the equivalent formulation

$$\frac{1}{\tilde{y} - y}\left(\left\{\frac{\ln(p(\Delta, \tilde{y}|\tilde{x})) - \ln(p(\Delta, \tilde{y}|x))}{\tilde{x} - x}\right\} - \left\{\frac{\ln(p(\Delta, y|\tilde{x})) - \ln(p(\Delta, y|x))}{\tilde{x} - x}\right\}\right) > 0. \tag{41}$$

The necessity of (15) then follows by taking the limit as $\tilde{x} \to x^+$ and $\tilde{y} \to y^+$ in (41). The sufficiency of (15) follows from the fact that a function whose derivative is positive is increasing: at a fixed x, $\partial \ln(p)/\partial x$ is an increasing function of y; hence, $\partial \ln(p(\Delta, \tilde{y}|x))/\partial x > \partial \ln(p(\Delta, y|x))/\partial x$. Thus $\{\ln(p(\Delta, \tilde{y}|x)) - \ln(p(\Delta, y|x))\}/\{\tilde{y} - y\}$ is an increasing function of x. Then (41), or equivalently (14), follows. Q.E.D.

Proof of Proposition 4: We want to show that, if condition (15) is satisfied by the transition density p_X of a process X, then the same condition is fulfilled by the transition density p_Y of the process $Y = \varphi^{-1}(X)$ where the function φ is twice continuously differentiable and strictly monotonic. Indeed the transition density of Y is given by the Jacobian formula:

$$\begin{aligned} p_Y(\Delta, y|y_0) &= \frac{\partial}{\partial y}\Pr(Y_{t+\Delta} \leq y | Y_t = y_0) = \frac{\partial}{\partial y}\Pr(X_{t+\Delta} \leq \varphi(y) | X_t = \varphi(y_0)) \\ &= \frac{\partial}{\partial y}\left[\int^{\varphi(y)} p_X(\Delta, x|\varphi(y_0))\, dx\right] = \varphi'(y) p_X(\Delta, \varphi(y)|\varphi(y_0)). \end{aligned} \tag{42}$$

Consequently, if X is a diffusion, then $\delta_X(\Delta, x|x_0) \equiv \partial^2 \ln(p(\Delta, y|x))/\partial x \partial y > 0$ from condition (15), and it follows from equation (42) that

$$\delta_Y(\Delta, y|y_0) \equiv \frac{\partial^2}{\partial y_0 \partial y}\ln(p_Y(\Delta, y|y_0)) = \varphi'(y)\varphi'(y_0)\delta_X(\Delta, \varphi(y)|\varphi(y_0)) > 0, \tag{43}$$

so the transition function of the Y process automatically satisfies condition (15). Q.E.D.

Proof of Proposition 5: With $J_0, J_1, \ldots$ denoting the jump times of the chain $X = \{X_t\}_{t\geq 0}$, and $S_1, S_2, \ldots$ its holding times, defined by

$$S_n = \begin{cases} J_n - J_{n-1} & \text{if } J_{n-1} < \infty \\ \infty & \text{otherwise,} \end{cases} \tag{44}$$

right-continuity forces $S_n > 0$ for all n. If $J_{n+1} = \infty$ for some n, define $X_\infty \equiv X_{J_n}$, otherwise X_∞ is undefined. For convenience, we set $X_t = \infty$ if t is greater than the first explosion time $\sup\{J_n/n \geq 0\} = \sum_{n=1}^{\infty} S_n$.

From the theory of continuous-time Markov chains (see, e.g., Norris (1997, page 87)), condition (18) on the transition matrix of the jump chain Y is equivalent to the restriction that the generator matrix A of the Markov chain X be of the Jacobi form: zero entries except on the diagonal, supradiagonal, and infradiagonal lines:

$$A = \begin{pmatrix} \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & \\ \cdots & 0 & & \beta_i & & -\alpha_i - \beta_i & & \alpha_i & & 0 & \cdots \\ & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots \end{pmatrix}, \tag{45}$$

where $\alpha_i > 0$, $\beta_i > 0$, and then $\lambda_i = \alpha_i/(\alpha_i + \beta_i)$. The element $[a_{ij}]$ of A determines the rate at which the chain moves from state i to state j.

To prove that (17) is equivalent to (45), first note from (2)—where the integral is replaced by a sum over all the possible intermediary states—that it suffices to prove the equivalence for an infinitesimal Δ, and the equivalence will then be carried forward in time by repeated use of (2). Next express the transition matrix $P(\Delta)$ in terms of the generator matrix A:

$$P(\Delta) = \exp[\Delta A] = \sum_{k=1}^{\infty} \frac{\Delta^k A^k}{k!}. \tag{46}$$

Suppose that A has a nonzero element outside the three lines indicated in (45), say $a_{i-1,i+1} = \gamma_{i-1} > 0$, with $a_{i-1,i-1} = -\alpha_{i-1} - \beta_{i-1} - \gamma_{i-1}$ now being the required diagonal term on the row. Then it follows from $P(\Delta) = I + \Delta A + o(\Delta)$ that (17) is violated since

$$\delta(\Delta, i, i+1 \mid i-1, i) = -\gamma_{i-1}\Delta + O(\Delta^2) < 0. \tag{47}$$

By contrast, if A has the form (45), then it follows from

$$P(\Delta) = I + \Delta A + \Delta^2 A^2/2 + o(\Delta^2) \tag{48}$$

that

$$\delta(\Delta, i, i+1 | i-1, i) = \alpha_i \alpha_{i-1} \Delta^2 - \alpha_i \alpha_{i-1} \Delta^2/2 + o(\Delta^2) = \alpha_i \alpha_{i-1} \Delta^2/2 \geq 0, \tag{49}$$

and similarly for the other transitions. The equivalence is therefore proved. Note that the form of the generator (45) characterizes birth and death processes, where the size of the population either goes up or down by one individual at a time. As a special case, this class includes Poisson processes for which $Y_n = n$ with probability one, so $\lambda_i = 1$ for all i. Q.E.D.

Proof of Proposition 6: Criterion (15) in the case of a transition density depending only on z reduces to $\partial^2 \ln(p(\Delta, z))/\partial z^2 < 0$, that is, log-concavity of the density. A large number of densities are log-concave. But one must remember that the process must be Markovian, that is, satisfy (2). The combination of both is enough to reduce the set of admissible models to the arithmetic Brownian motion.

Indeed, consider the leading term at order Δ^{-1} of the expansion of the transition density of such a space-homogeneous diffusion:

$$\begin{aligned} \ln(\tilde{p}_X^{(-1)}(\Delta, x_0 + z | x_0)) = & -\frac{1}{2}\ln(2\pi\Delta) - \ln(\sigma(x_0 + z)) \\ & - \frac{1}{2\Delta}(\gamma(x_0 + z) - \gamma(x_0))^2 \end{aligned} \tag{50}$$

(see Aït-Sahalia (1999, 2002)). We are asking when this function depends on z but not on x_0. By a Taylor expansion in z around zero, the right-hand side of equation (50) is independent of x_0 if and only if the function σ is, in which case $\gamma(x) = x/\sigma$. Looking then at the next order term,

$$\ln(\tilde{p}_X^{(0)}(\Delta, x_0 + z | x_0)) - \ln(\tilde{p}_X^{(-1)}(\Delta, x_0 + z | x_0)) = \int_{\gamma(x_0)}^{\gamma(x_0+z)} \mu_Y(v)\, dv, \tag{51}$$

where

$$\mu_Y(y) = \mu(\gamma^{-1}(y))/\sigma(\gamma^{-1}(y)) - \sigma'(\gamma^{-1}(y))/2. \tag{52}$$

Given that γ is linear, the right-hand side of equation (51) can only be independent of x_0 if $\mu_Y(\cdot)$ is constant. From equation (52) and $\sigma(x) = \sigma$, $\gamma(x) = x/\sigma$, this can only occur if $\mu(x) = \mu$ is constant. Hence, the only process with space-homogeneous transition density is the arithmetic Brownian motion with $\sigma(x) = \sigma$ and $\mu(x) = \mu$ both constant. Q.E.D.

Proof of Proposition 7: First note that if $(X_{t+\Delta}, X_t)$ has a Gaussian distribution, then so does $X_{t+\Delta}|X_t$, and, moreover, $E[X_{t+\Delta}|X_t]$ must be affine in X_t and $V[X_{t+\Delta}|X_t]$ constant in X_t. This follows, for instance, from Theorem III.6.5, page 86, of Feller (1971). Let us therefore consider Gaussian transition functions $p(\Delta, y|x)$ with conditional mean $e_\Delta(x) \equiv E[X_{t+\Delta}|X_t = x] = \alpha_\Delta + \beta_\Delta x$ and conditional variance $v_\Delta \equiv V[X_{t+\Delta}|X_t = x]$, and let us see what further restrictions on the dependence of $\alpha_\Delta, \beta_\Delta$, and v_Δ on Δ make these functions compatible with conditions (2) for Markovianity and (15) for diffusion. That is, should further restrictions on the conditional mean and variance be placed to insure that the discrete observations are embeddable in a diffusion?

It turns out that in this case, the Markov requirement alone is sufficient to reduce the set of compatible transition densities to the Ornstein–Uhlenbeck case which we already know is a diffusion.[15] In other words, there exists no nondiffusion Markov process with Gaussian transitions. Equivalently, this means that for Gaussian transitions once the Markov requirement is imposed, diffusion condition (15) is automatically satisfied. Indeed, the Chapman–Kolmogorov equation (2) implies by a direct calculation that

$$\alpha_{2\Delta} = \alpha_\Delta(1 + \beta_\Delta), \beta_{2\Delta} = \beta_\Delta^2, v_{2\Delta} = v_\Delta(1 + \beta_\Delta^2).$$

With $\alpha_0 = 0$, $\beta_0 = 1$, and $v_0 = 0$, the only solution is of the form

$$\alpha_\Delta = \alpha(1 - e^{-\beta\Delta}), \beta_\Delta = e^{-\beta\Delta}, v_\Delta = \frac{v}{2\beta}(1 - e^{-2\beta\Delta}),$$

where α, β and $v > 0$ are constants. Applying now condition (15) to

$$p(\Delta, y|x) = (2\pi v_\Delta)^{-1/2} \exp\{-(y - e_\Delta(x))^2/(2v_\Delta)\}$$

yields $\partial^2 \ln(p(\Delta, y|x))/\partial x \partial y = \beta_\Delta / v_\Delta > 0$. Therefore, the diffusion condition puts no additional constraints on the model's parameters in the Gaussian case, so saying that the process is Markovian and Gaussian is enough to reduce the admissible set to the Ornstein–Uhlenbeck class. Q.E.D.

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[15] Note that this includes the arithmetic Brownian motion as the special case where $\beta = 0$.

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